5. Dimensional regularization and renormalization

φ^4 – theory, one-loop level:

$$m_{ph}^2 = m^2 - \frac{i\lambda}{2}\Delta(0) + \mathcal{O}(\lambda^2)$$

interaction changes the mass of the particle: bare mass m (parameter in $\mathcal{L}$) differs from the physical mass m_{ph} by a correction of $\mathcal{O}(\lambda)$ (further shifts at higher orders)

remark: $\Sigma'(k^2) = 0$ at one-loop level $\Rightarrow Z = 1 + \mathcal{O}(\lambda^2)$; k^2-dependence of $\Sigma(k^2)$ starts only at two loops

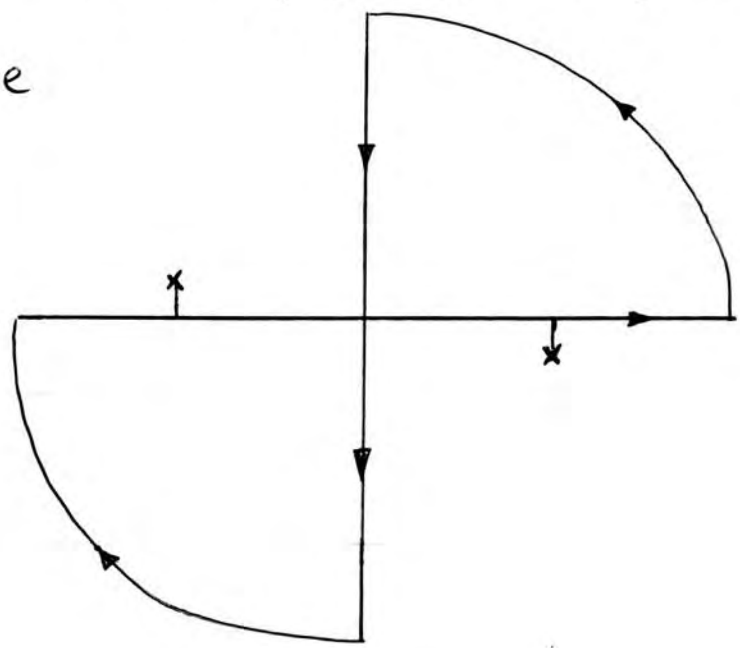
what is the form of $\Delta(0)$ in dimensional regularization?

basic idea: regularization of the propagator by analytic continuation in the dimension d of space-time at the origin, the propagator is a singular object → regularization is essential

$$\Delta(0) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{m^2 - k^2 - i\varepsilon}$$

integral <u>converges</u> for $d < 2$ → defines an <u>analytic function of d</u> → admits an unambiguous <u>continuation</u> in the analyticity domain (only the above integral representation of this function is limited to $d<2$)

poles at $k^0 = \pm\left(\sqrt{m^2 + \vec{k}^{\,2}} - i\varepsilon\right)$ in complex k^0-plane



turn the contour of integration in counterclockwise direction from the real to the imaginary axis

$$\rightarrow \qquad \Delta(0) = i \int \frac{d^d k}{(2\pi)^d} \; \frac{1}{m^2 + k^2}$$

where k^2 stands now for the <u>euclidean</u> square $\quad k^2 = (k^0)^2 + \ldots + (k^d)^2$

use $$\int_0^\infty dt \; e^{-tA} = \frac{1}{A} \quad , \; A > 0$$

$$\Rightarrow \quad \Delta(0) = i \int_0^\infty dt \; e^{-tm^2} \underbrace{\int \frac{d^d k}{(2\pi)^d} \; e^{-tk^2}}_{(2\pi)^{-d} \left(\frac{\pi}{t}\right)^{d/2}}$$

$$= \frac{i}{(4\pi)^{d/2}} \int_0^\infty dt \; e^{-tm^2} \; t^{-d/2} \underset{tm^2 = u}{\overset{\uparrow}{=}}$$

$$= \frac{i \; m^{d-2}}{(4\pi)^{d/2}} \int_0^\infty du \; e^{-u} \; u^{-d/2}$$

$$= \frac{i \; m^{d-2}}{(4\pi)^{d/2}} \; \Gamma\left(1 - \frac{d}{2}\right)$$

$\Delta(0)$ is an analytic function of d, except for poles at $d = 2, 4, 6, \dots$

behaviour of $\Delta(0)$ in the vicinity of $d = 4$:

$$\left.\begin{aligned} x\,\Gamma(x) &= \Gamma(x+1) \\ (x+1)\,\Gamma(x+1) &= \Gamma(x+2) \end{aligned}\right\} \Rightarrow \; x\,(x+1)\,\Gamma(x) = \Gamma(x+2)$$

$$\Rightarrow \quad \Gamma\left(1-\frac{d}{2}\right) = \frac{\Gamma\left(3-\frac{d}{2}\right)}{\left(1-\frac{d}{2}\right)\left(2-\frac{d}{2}\right)}$$

$$= \frac{2}{d-4} - \Gamma'(1) + \mathcal{O}(d-4)$$

back to φ^4-theory:

$$m_{ph}^2 = m^2 + \frac{\lambda}{2}\,\frac{m^{d-2}}{(4\pi)^{d/2}}\,\Gamma\left(1-\frac{d}{2}\right) + \mathcal{O}(\lambda^2)$$

<u>observation:</u> m fix, $d \to 4$ would lead to divergent m_{ph} (nonsensical result)

<u>instead:</u> $m = m(d, m_{ph})$ such that m_{ph} <u>finite</u> for $d \to 4$ (<u>mass renormalization</u>)

inverting the above relation gives

$$m^2 = m_{ph}^2 - \frac{\lambda}{2} \frac{m_{ph}^{d-2}}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) + \mathcal{O}(\lambda^2)$$

remark: $\lambda\,(m_{ph}^2 - m^2) = \mathcal{O}(\lambda^2)$ was used

bare mass occurs as a parameter in the Lagrangian that defines the model, but it does <u>not</u> represent a physical quantity

Lagrangian only makes sense if it is supplemented with a <u>regularization</u>

value of m <u>depends</u> on the regularization scheme

<u>only</u> m_{ph} is <u>observable</u>

also the coupling constant λ is an unphysical parameter $\rightarrow$ will define λ_{ph} related to observable scattering amplitude of the theory (see later) computation of a typical <u>one-loop integral</u> in dimensional regularization

$$I = \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{(M^2 - k^2 - i\varepsilon)^\alpha} \qquad (\alpha \text{ integer})$$

→ <u>euclidean</u> integral (convergent for $d < 2\alpha$)

$$I = i \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{(M^2 + k^2)^\alpha}$$

we use the formula

$$\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-1} \, e^{-tA} \, , \quad A > 0$$

which follows from our previous formula

$$\frac{1}{A} = \int_0^\infty dt \, e^{-tA}$$

by differentiating with respect to A

$$\Rightarrow \quad I = \frac{i}{\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-1} e^{-tM^2} \underbrace{\int \frac{d^d k}{(2\pi)^d} \, e^{-tk^2}}_{\frac{1}{(4\pi^2)^{d/2}} \left(\frac{\pi}{t}\right)^{d/2}}$$

$$= \frac{i}{(4\pi)^{d/2}\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-1-\frac{d}{2}} e^{-tM^2} \underset{tM^2 = u}{\overset{\uparrow}{=}}$$

$$= \frac{i \, M^{d-2\alpha}}{(4\pi)^{d/2}\,\Gamma(\alpha)} \, \Gamma\left(\alpha - \frac{d}{2}\right)$$

remark: $\alpha = 1 \to$ previous result

<u>exercise:</u>

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^\beta}{(M^2 - k^2 - i\varepsilon)^\alpha} = \frac{(-1)^\beta i}{(4\pi)^{d/2}} \frac{\Gamma(\alpha-\beta-\frac{d}{2})\,\Gamma(\beta+\frac{d}{2})}{\Gamma(\alpha)\,\Gamma(\frac{d}{2})} M^{d+2\beta-2\alpha}$$

α, β integer

<u>important property:</u> $\alpha = 0 \Rightarrow \int \frac{d^d k}{(2\pi)^d} (k^2)^\beta = 0$

for arbitrary β, e.g. $\int \frac{d^d k}{(2\pi)^d} = 0$ in dim. reg.

$$\int \frac{d^d k}{(2\pi)^d} e^{ikx} = \delta^{(d)}(x) \Rightarrow \delta^{(d)}(0) = \int \frac{d^d k}{(2\pi)^d} = 0$$

Fourier transform of <u>connected</u> n-point Green's function:

$$\int d^dx_1 \ldots d^dx_n \, e^{ik_1x_1} \ldots e^{ik_nx_n} \langle 0 | T \phi(x_1) \ldots \phi(x_n) | 0 \rangle_c$$

$$\underset{\substack{\uparrow \\ \text{translation} \\ \text{invariance}}}{=} (2\pi)^d \underbrace{\delta^{(d)}(k_1 + \ldots + k_n)}_{\substack{\text{energy-momentum} \\ \text{conservation}}} \Gamma_n(k_1, \ldots, k_n)$$

coefficient $\Gamma_n(k_1, \ldots, k_n)$ only defined for conserved energies and momenta $(k_1 + \ldots + k_n = 0)$

perturbation series for $\Gamma_n(k_1, \ldots, k_n)$ in φ^4-theory:

$$(2\pi)^d \delta^{(d)}(k_1 + \ldots + k_n) \Gamma_n(k_1, \ldots, k_n)$$

$$= \int d^dx_1 \ldots d^dx_n \, e^{ik_1x_1} \ldots e^{ik_nx_n}$$

$$\langle\langle \varphi(x_1) \ldots \varphi(x_n) \, e^{iS_{int}} \rangle\rangle_c$$

$$= \langle\langle \tilde{\varphi}(k_1) \ldots \tilde{\varphi}(k_n) \, e^{iS_{int}} \rangle\rangle_c$$

with $\int d^dx \, \varphi(x) \, e^{ikx} = \tilde{\varphi}(k)$

$$\Rightarrow \quad (2\pi)^d\, \delta^{(d)}(k_1+\dots+k_n)\ \Gamma_n(k_1,\dots,k_n) =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{i\lambda}{4!}\right)^k \int d^dy_1 \dots d^dy_k$$

$$\left\langle\!\left\langle \tilde\varphi(k_1)\dots\tilde\varphi(k_n)\,\varphi(y_1)^4 \dots \varphi(y_k)^4 \right\rangle\!\right\rangle_c$$

only connected parts relevant $\Rightarrow$ $\tilde\varphi(k)$ must be paired with $\varphi(y)$:

$$\langle\!\langle \tilde\varphi(k)\,\varphi(y)\rangle\!\rangle = \int d^dx\, e^{ikx}\, \underbrace{\langle\!\langle \varphi(x)\,\varphi(y)\rangle\!\rangle}_{\frac{1}{i}\Delta(x-y)}$$

$$= \int d^dx\, e^{ikx}\, \frac{1}{i}\int \frac{d^d\ell}{(2\pi)^d}\, \frac{e^{-i\ell(x-y)}}{m^2-\ell^2-i\varepsilon}$$

$$= \frac{1}{i}\int \frac{d^d\ell}{(2\pi)^d}\,(2\pi)^d\,\delta^{(d)}(k-\ell)\, e^{i\ell y}\, \frac{1}{m^2-\ell^2-i\varepsilon}$$

$$= \frac{1}{i}\, \frac{e^{iky}}{m^2-k^2-i\varepsilon}$$

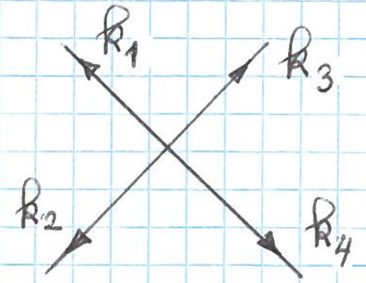
internal lines: $\langle\!\langle \varphi(y)\,\varphi(y')\rangle\!\rangle = \int \frac{d^dq}{(2\pi)^d}\,\frac{1}{i}\,\frac{e^{-iq(y-y')}}{m^2-q^2-i\varepsilon}$

four-point function

tree contribution:

$$(2\pi)^d \, \delta^{(d)}(k_1+k_2+k_3+k_4) \; \Gamma_4(k_1,k_2,k_3,k_4) =$$

$$= \frac{-i\lambda}{4!} \int d^d y \, \left\langle\!\left\langle \tilde{\varphi}(k_1) \ldots \tilde{\varphi}(k_n) \, \varphi(y)^4 \right\rangle\!\right\rangle_c = (*)$$



$4 \times 3 \times 2 = 4!$ contributions from pairing rule

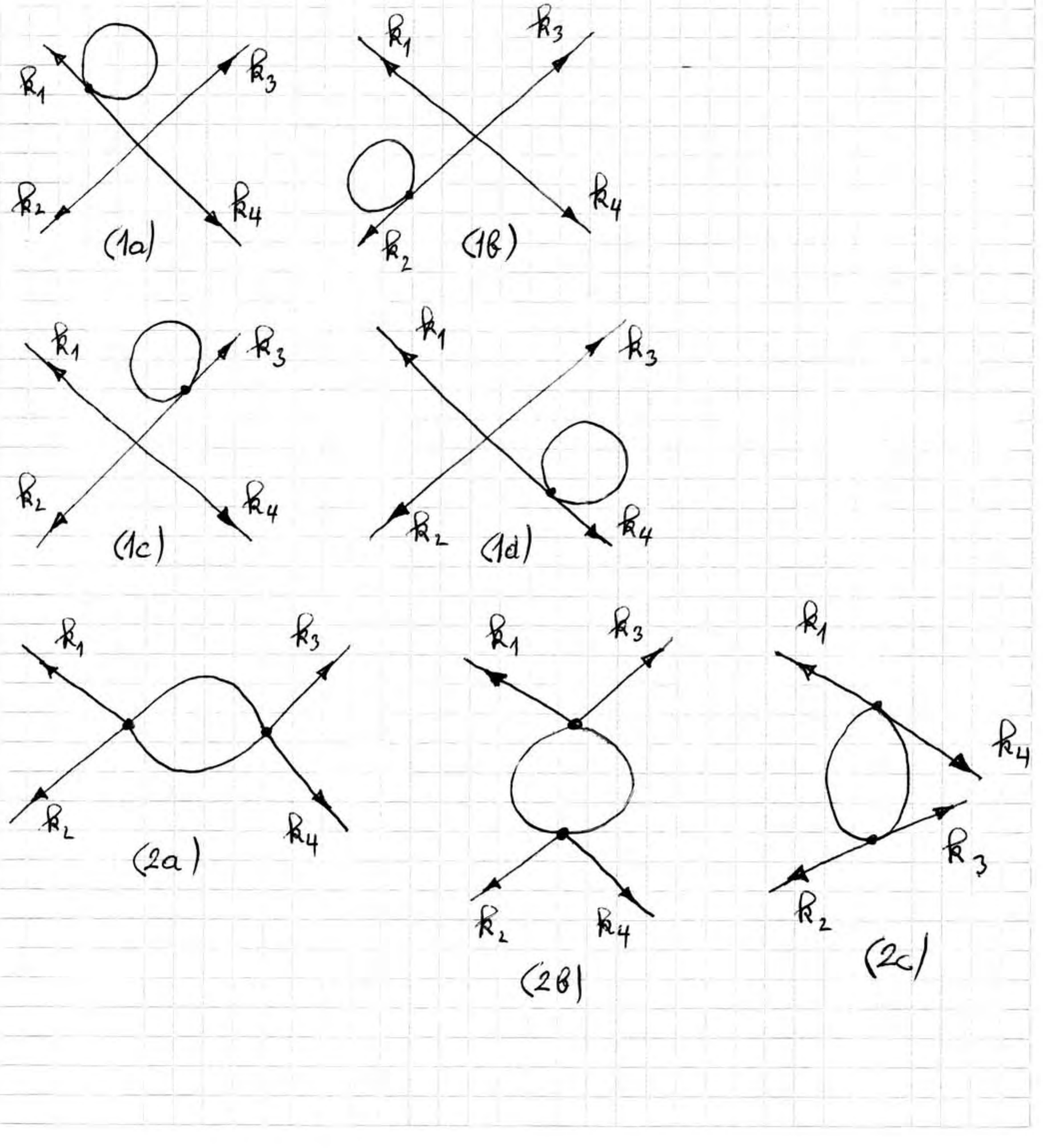
$$\Rightarrow (*) = \frac{-i\lambda}{4!} \left(\frac{1}{i}\right)^4 \frac{4! \, (2\pi)^d \, \delta^{(d)}(k_1+k_2+k_3+k_4)}{(m^2-k_1^2-i\varepsilon)(m^2-k_2^2-i\varepsilon)(m^2-k_3^2-i\varepsilon)(m^2-k_4^2-i\varepsilon)}$$

$$\Rightarrow \Gamma_4(k_1,k_2,k_3,k_4) = \frac{-i\lambda}{i(m^2-k_1^2-i\varepsilon)\, i(m^2-k_2^2-i\varepsilon)\, i(m^2-k_3^2-i\varepsilon)\, i(m^2-k_4^2-i\varepsilon)}$$

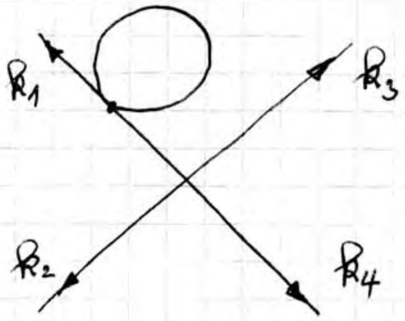
remark: scattering amplitude (at tree-level) just given by $-\lambda$ (see later) one-loop contributions:

$$\frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2 \int d^d y_1\, d^d y_2 \left\langle\!\!\left\langle \tilde{\varphi}(k_1)\ldots\tilde{\varphi}(k_4)\, \varphi(y_1)^4 \varphi(y_2)^4 \right\rangle\!\!\right\rangle_c$$

diagrams:



we consider the first diagram:



multiplicity factor:

$$4 \times 3 \times 4 \times 3! \times 2 = (4!)^2$$

$$\Rightarrow \quad (2\pi)^d \, \delta^{(d)}(k_1+k_2+k_3+k_4) \, \Gamma_4(k_1, k_2, k_3, k_4)\Big|_{1a} =$$

$$= (4!)^2 \frac{1}{2!} \left(-\frac{i\lambda}{4!}\right)^2 \int d^d y_1 \, d^d y_2 \, \frac{1}{i} \frac{e^{ik_1 y_1}}{m^2 - k_1^2 - i\varepsilon}$$

$$\times \frac{1}{i} \frac{e^{ik_2 y_2}}{m^2 - k_2^2 - i\varepsilon} \; \frac{1}{i} \frac{e^{ik_3 y_2}}{m^2 - k_3^2 - i\varepsilon} \; \frac{1}{i} \frac{e^{ik_4 y_2}}{m^2 - k_4^2 - i\varepsilon}$$

$$\times \frac{1}{i} \Delta(0) \, \frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{e^{-iq(y_1 - y_2)}}{m^2 - q^2 - i\varepsilon}$$

$$= \frac{\lambda^2}{2!} \Delta(0) \int \frac{d^d q}{(2\pi)^d} (2\pi)^d \delta^{(d)}(k_1 - q) \, (2\pi)^d \delta^{(d)}(k_2 + k_3 + k_4 + q)$$

$$\times \frac{1}{m^2 - k_1^2 - i\varepsilon} \; \frac{1}{m^2 - k_2^2 - i\varepsilon} \; \frac{1}{m^2 - k_3^2 - i\varepsilon} \; \frac{1}{m^2 - k_4^2 - i\varepsilon}$$

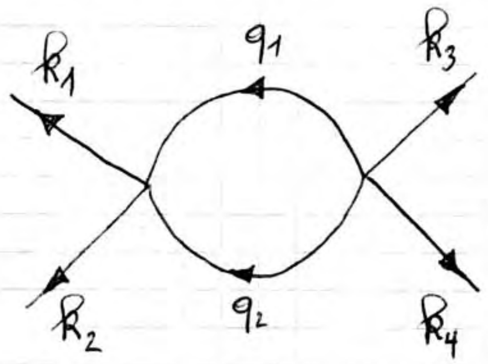
$$\times \frac{1}{m^2 - q^2 - i\varepsilon}$$

$$= (2\pi)^d \, \delta^{(d)}(k_1 + k_2 + k_3 + k_4)$$

$$\underbrace{\times \frac{\lambda^2 \Delta(0)}{2!} \frac{1}{(m^2 - k_1^2 - i\varepsilon)^2 \, (m^2 - k_2^2 - i\varepsilon)\,(m^2 - k_3^2 - i\varepsilon)\,(m^2 - k_4^2 - i\varepsilon)}}_{\Gamma_4(k_1, k_2, k_3, k_4)\big|_{1a}}$$

contributions from (1b), (1c), (1d) obtained by $k_1 \leftrightarrow k_2, k_3, k_4$

diagram (2a):



multiplicity factor:

$$4 \times 3 \times 4 \times 3 \times 2 \times 2 = (4!)^2$$

$$(2\pi)^d \, \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \, \Gamma(k_1, k_2, k_3, k_4)\big|_{2a}$$

$$= (4!)^2 \frac{1}{2!} \left(-\frac{i\lambda}{4!}\right)^2 \frac{1}{i} \frac{1}{m^2 - k_1^2 - i\varepsilon} \cdots \frac{1}{i} \frac{1}{m^2 - k_2^2 - i\varepsilon}$$

$$\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{i} \frac{1}{m^2 - q_1^2 - i\varepsilon} \frac{1}{i} \frac{1}{m^2 - q_2^2 - i\varepsilon}$$

$$(2\pi)^d \, \delta^{(d)}(q_1 + q_2 - k_1 - k_2) \quad (2\pi)^d \, \delta^{(d)}(q_1 + q_2 + k_3 + k_4)$$

$$= (2\pi)^d \, \delta^{(d)}(k_1 + k_2 + k_3 + k_4)$$

$$\times \frac{\lambda^2}{2} \frac{1}{(m - k_1^2 - i\varepsilon)(m^2 - k_2^2 - i\varepsilon)(m^2 - k_3^2 - i\varepsilon)(m^2 - k_4^2 - i\varepsilon)}$$

$$\times \int \frac{d^d q}{(2\pi)^d} \frac{1}{m^2 - q^2 - i\varepsilon} \frac{1}{m^2 - (q - k_1 - k_2)^2 - i\varepsilon}$$

one-loop integral

$$i\,B(p^2, m^2) := \int \frac{d^d q}{(2\pi)^d} \frac{1}{m^2 - q^2 - i\varepsilon} \frac{1}{m^2 - (q-p)^2 - i\varepsilon}$$

$$\Gamma_4(k_1, k_2, k_3, k_4)\Big|_{2a} = \frac{\frac{i\lambda^2}{2} B(s, m^2)}{(m^2 - k_1^2 - i\varepsilon)(m^2 - k_2^2 - i\varepsilon)(m^2 - k_3^2 - i\varepsilon)(m^2 - k_4^2 - i\varepsilon)}$$

where $s = (k_1 + k_2)^2$ (Mandelstam variable)

diagram (2b): $s \to t = (k_1 + k_3)^2$

diagram (2c): $s \to u = (k_1 + k_4)^2$

we want to eliminate the bare mass m in favour of the physical mass m_{ph}

$$m_{ph}^2 = m^2 - \frac{i\lambda}{2}\Delta(0) + O(\lambda^2)$$

external line factors expressed in terms of m_{ph}:

$$\frac{1}{m^2 - k^2} = \frac{1}{m_{ph}^2 + \frac{i\lambda}{2}\Delta(0) - k^2 + O(\lambda^2)} =$$

$$= \frac{1}{m_{ph}^2 - k^2} - \frac{\frac{i}{2}\lambda\Delta(0)}{(m_{ph}^2 - k^2)^2} + O(\lambda^2)$$

$$\Gamma_4\Big|_{tree} + \Gamma_4\Big|_{1a+1b+1c+1d} =$$

$$-i\lambda \left[\frac{1}{m_{ph}^2 - k_1^2} - \frac{\frac{i}{2}\lambda\Delta(0)}{(m_{ph}^2 - k_1^2)^2}\right]$$

$$\times \left[\frac{1}{m_{ph}^2 - k_2^2} - \frac{\frac{i}{2}\lambda\Delta(0)}{(m_{ph}^2 - k_2^2)^2}\right]$$

$$\times \left[\frac{1}{m_{ph}^2 - k_3^2} - \frac{\frac{i}{\lambda}\Delta(0)}{(m_{ph}^2 - k_3^2)^2}\right]$$

$$\times \left[\frac{1}{m_{ph}^2 - k_4^2} - \frac{\frac{i}{\lambda}\Delta(0)}{(m_{ph}^2 - k_4^2)^2}\right]$$

$$+ \frac{\lambda^2 \Delta(0)}{2} \left[\frac{1}{m^2 - k_1^2} + \frac{1}{m^2 - k_2^2} + \frac{1}{m^2 - k_3^2} + \frac{1}{m^2 - k_4^2} \right]$$

$$\times \frac{1}{m^2 - k_1^2} \frac{1}{m^2 - k_2^2} \frac{1}{m^2 - k_3^2} \frac{1}{m^2 - k_4^2} + O(\lambda^3)$$

$$= \frac{-i\lambda}{(m_{ph}^2 - k_1^2)(m_{ph}^2 - k_2^2)(m_{ph}^2 - k_3^2)(m_{ph}^2 - k_4^2)}$$

effect of external line insertions: $m \longrightarrow m_{ph}$

$$\lambda^2 B(s, m^2) = \lambda^2 B(s, m_{ph}^2) + O(\lambda^3)$$

$\Rightarrow$ final result:

$$\Gamma_4(k_1, k_2, k_3, k_4) = -i \frac{\lambda - \frac{\lambda^2}{2}\left[B(s, m_{ph}^2) + B(t, m_{ph}^2) + B(u, m_{ph}^2)\right]}{(m_{ph}^2 - k_1^2)(m_{ph}^2 - k_2^2)(m_{ph}^2 - k_3^2)(m_{ph}^2 - k_4^2)}$$

$$+ O(\lambda^3)$$

_properties of the loop function $B(p^2; m^2)$:

$$i\,B(p^2, m^2) = \int \frac{d^dq}{(2\pi)^d} \frac{1}{m^2 - q^2 - i\varepsilon} \frac{1}{m^2 - (q-p)^2 - i\varepsilon}$$

trick of Schwinger and Feynman:

$$\frac{1}{A_1 A_2} = \int_0^1 d\alpha \frac{1}{[(1-\alpha)A_1 + \alpha A_2]^2}$$

remark: general case of n factors:

$$\frac{1}{A_1 A_2 \cdots\cdots A_n} = (n-1)! \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n\, \delta(1-\alpha_1 - \ldots - \alpha_n)$$

$$\times \frac{1}{[\alpha_1 A_1 + \ldots\ldots + \alpha_n A_n]^n}$$

$$\Rightarrow\ i\,B(p^2; m^2) = \int_0^1 d\alpha \int \frac{d^dq}{(2\pi)^d} \frac{1}{[(1-\alpha)(m^2-q^2-i\varepsilon) + \alpha(m^2-(q-p)^2-i\varepsilon)]^2}$$

$$(1-\alpha)(m^2 - q^2 - i\varepsilon) + \alpha\,(m^2 - (q-p)^2 - i\varepsilon) =$$

$$= m^2 - q^2 + 2\alpha\, p.q - \alpha p^2 - i\varepsilon$$

$$= m^2 - \underbrace{(q - \alpha p)}_{k}{}^2 - \alpha(1-\alpha)p^2 - i\varepsilon$$

$$\Rightarrow \; i\,B(p^2, m^2) = \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \; \frac{1}{\left[m^2 - k^2 - \alpha(1-\alpha)p^2 - i\varepsilon\right]^2}$$

$$= \frac{i\,\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 d\alpha \; \left[m^2 - \alpha(1-\alpha)p^2\right]^{\frac{d-4}{2}}$$

see p. 5/7

$$\xrightarrow[d\to 4]{} \; \underbrace{-\frac{i}{8\pi^2(d-4)}}_{\text{independent of } p^2} + \text{finite terms}$$

$\Rightarrow \; \overline{B}(p^2, m^2) := B(p^2, m^2) - B(0, m^2)$ is a <u>convergent</u> integral

$\Rightarrow$ <u>divergent</u> part of

$$\Gamma_4(k_1, \dots, k_4) = -i \; \frac{\lambda - \frac{\lambda^2}{2}\left[B(s, m_{ph}^2) + B(t, m_{ph}^2) + B(u, m_{ph}^2)\right]}{(m_{ph}^2 - k_1^2) \;\; \dots \;\; (m_{ph}^2 - k_4^2)}$$

has the <u>same structure</u> as the tree graph contribution

$\rightarrow$ change in the value of λ

in contrast to the case of the mass, where the physical value is uniquely determined by the energy-momentum relation, the definition of the physical coupling constant is <u>convention dependent</u>

will see later: <u>scattering amplitude</u> $\mathcal{M}(p_1, p_2 \to p_3, p_4)$ for the process $p_1 p_2 \to p_3 p_4$ (on-shell momenta), defined by

$$\langle p_3, p_4 \text{ out} \,|\, p_1, p_2 \text{ in} \rangle = i\,(2\pi)^d\, \delta^{(d)}(p_1 + p_2 - p_3 - p_4) \times \mathcal{M}(p_1, p_2 \to p_3, p_4),$$

related to $\Gamma_4(k_1, k_2, k_3, k_4)$ by the following procedure (LSZ formalism): remove the external propagators from Γ_4, divide by iZ^2 (in our case: $Z = 1 + O(\lambda^3)$), and put

$k_1 = -p_1$, $k_2 = -p_2$, $k_3 = p_3$, $k_4 = p_4$ (on mass-shell)

$$\Rightarrow \mathcal{M}(p_1, p_2 \to p_3, p_4) = -\lambda + \frac{\lambda^2}{2}\left[B(s, m_{ph}^2) + B(t, m_{ph}^2) + B(u, m_{ph}^2)\right]$$

with $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$

remark: $s+t+u = 6m_{ph}^2 + 2p_1 \underbrace{(p_2 - p_3 - p_4)}_{-p_1}$

$$= 4m_{ph}^2$$

→ only <u>two</u> independent Lorentz invariants (e.g. s,t)

in the <u>center of mass system</u> (CMS): $\sqrt{s}$ = total energy of incoming particles

$p_1 = (E, \vec{p})$, $p_2 = (E, -\vec{p})$ $\Rightarrow s = 4E^2$

$\Rightarrow \sqrt{s} = 2E$

t, u can be related to the scattering angle θ by

$$t = -\frac{1}{2}(s - 4m_{ph}^2)(1-\cos\theta)$$

$$u = -\frac{1}{2}(s - 4m_{ph}^2)(1+\cos\theta)$$

$$\mathcal{M}(p_1, p_2 \to p_3, p_4) = \mathcal{M}(s,t)$$

$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}(s,t)|^2$ <u>differential cross section</u> (observable)

$$\underbrace{\mathcal{M}(s,t)}_{\text{finite}} = -\lambda \left\{ 1 - \frac{\lambda}{2} \left[B(s,m_{ph}^2) + B(t,m_{ph}^2) + B(u,m_{ph}^2) \right] + \mathcal{O}(\lambda^2) \right\}$$

$$B(s, m_{ph}^2) = -\frac{1}{8\pi^2(d-4)} + \text{finite terms}$$

λ must be chosen in such a way that the divergences in the B-functions are compensated in the limit $d \to 4$

$$|\mathcal{M}(s,t)|^2 = \lambda^2 \left\{ 1 - \lambda \,\mathrm{Re} \left[B(s,m_{ph}^2) + B(t,m_{ph}^2) + B(u,m_{ph}^2) \right] + \mathcal{O}(\lambda^2) \right\}$$

$$\Rightarrow |\mathcal{M}(s,t)| = \lambda \left\{ 1 - \frac{\lambda}{2} \mathrm{Re} \left[B(s,m_{ph}^2) + B(t,m_{ph}^2) + B(u,m_{ph}^2) \right] + \mathcal{O}(\lambda^2) \right\}$$

possible definition of <u>physical</u> coupling constant:

$$\lambda_{ph} := \underbrace{|\mathcal{M}(s_0,t_0)|}_{\substack{\text{measured at physically} \\ \text{allowed values } s_0, t_0}} = \lambda \left\{ 1 - \frac{\lambda}{2} \mathrm{Re} \left[B(s_0,m_{ph}^2) + B(t_0,m_{ph}^2) + B(u_0,m_{ph}^2) \right] + \mathcal{O}(\lambda^2) \right\}$$

$$u_0 = 4m_{ph}^2 - s_0 - t_0$$

$$\Rightarrow \lambda = \lambda_{ph} \left\{ 1 + \frac{\lambda_{ph}}{2} \mathrm{Re} \left[B(s_0,m_{ph}^2) + B(t_0,m_{ph}^2) + B(u_0,m_{ph}^2) \right] + \mathcal{O}(\lambda_{ph}^2) \right\}$$

express scattering amplitude in terms of λ_{ph}:

$$-\mathcal{M}(s,t) = \lambda_{ph}\left\{1 + \frac{\lambda_{ph}}{2}\,\mathrm{Re}\left[B(s_0, m_{ph}^2) + B(t_0, m_{ph}^2) + B(u_0, m_{ph}^2)\right] + \mathcal{O}(\lambda_{ph}^2)\right\}$$

$$\times\left\{1 - \frac{\lambda_{ph}}{2}\left[B(s, m_{ph}^2) + B(t, m_{ph}^2) + B(u, m_{ph}^2)\right] + \mathcal{O}(\lambda_{ph}^2)\right\}$$

$$= \lambda_{ph}\left\{1 - \frac{\lambda_{ph}}{2}\left[\underbrace{B(s, m_{ph}^2) - \mathrm{Re}B(s_0, m_{ph}^2)}_{\text{finite}}\right.\right.$$

$$\left.\left. + \underbrace{B(t, m_{ph}^2) - \underbrace{B(t_0, m_{ph}^2)}_{\substack{= \mathrm{Re}B(t_0, m_{ph}^2) \\ \text{as } t_0 \le 0}}}_{\text{finite}} + \underbrace{B(u, m_{ph}^2) - B(u_0, m_{ph}^2)}_{\text{finite}}\right] + \mathcal{O}(\lambda_{ph}^2)\right\}$$

<u>another possibility</u>: define $\tilde{\lambda}_{ph} := -\mathcal{M}(s_1, t_1)$

at the unphysical point $s_1 = t_1 = u_1 = \frac{4}{3} m_{ph}^2$

$$\Rightarrow \tilde{\lambda}_{ph} = \lambda - \frac{3}{2}\lambda^2 B(s_1, m_{ph}^2)$$

exercise: determine the relation between λ_{ph} and $\tilde{\lambda}_{ph}$