4. Perturbation theory

theory with interaction term:

$$S[\varphi] = S_0[\varphi] + S_{int}[\varphi]$$

action of free theory
(bilinear in φ)

example: φ^4 - theory

$$S_{int}[\varphi] = -\frac{\lambda}{4!}\int d^dx\, \varphi(x)^4$$

remark: anticipate already dimensional regularization ($d^4x \to d^dx$)

- perturbation series:

$$e^{iS} = e^{iS_0}\, e^{iS_{int}} = e^{iS_0} \sum_{k=0}^{\infty} \frac{i^k}{k!} S_{int}^k$$

vacuum of interacting theory

$$\langle 0 | T\, \phi(x_1) \dots \phi(x_n) | 0 \rangle =$$

$$= \frac{\int [d\varphi]\, e^{iS}\, \varphi(x_1) \dots \varphi(x_n)}{\int [d\varphi]\, e^{iS}} =$$

$$= \frac{\int [d\varphi]\, e^{iS_0[\varphi]} e^{iS_{int}[\varphi]} \varphi(x_1)\dots\varphi(x_n)}{\int [d\varphi]\, e^{iS_0[\varphi]}} \cdot \frac{\int [d\varphi]\, e^{iS_0[\varphi]}}{\int [d\varphi]\, e^{iS_0[\varphi]} e^{iS_{int}[\varphi]}}$$

$$= \frac{\left\langle\!\left\langle e^{iS_{int}[\varphi]} \varphi(x_1)\dots\varphi(x_n) \right\rangle\!\right\rangle}{\left\langle\!\left\langle e^{iS_{int}[\varphi]} \right\rangle\!\right\rangle}$$

where $\left\langle\!\left\langle \mathcal{F}[\varphi] \right\rangle\!\right\rangle := \dfrac{\int [d\varphi]\, e^{iS_0[\varphi]}\, \mathcal{F}[\varphi]}{\int [d\varphi]\, e^{iS_0[\varphi]}}$

Gaussian mean value of the function $\mathcal{F}[\varphi]$

perturbative expansion:

$$\left\langle\!\left\langle e^{iS_{int}[\varphi]} \varphi(x_1)\dots\varphi(x_n) \right\rangle\!\right\rangle =$$

$$= \sum_{k=0}^{\infty} \frac{i^k}{k!} \left\langle\!\left\langle S_{int}[\varphi]^k\, \varphi(x_1)\dots\varphi(x_n) \right\rangle\!\right\rangle$$

in the case of a φ^4 - theory:

$$\left\langle\!\left\langle e^{i S_{int}[\varphi]} \varphi(x_1) \ldots \varphi(x_n) \right\rangle\!\right\rangle =$$

$$= \sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{k!\,(4!)^k} \int d^d y_1 \ldots d^d y_k \left\langle\!\left\langle \varphi(y_1)^4 \ldots \varphi(y_k)^4 \varphi(x_1) \ldots \varphi(x_n) \right\rangle\!\right\rangle$$

we know already:

$$\left\langle\!\left\langle \varphi(x_1) \ldots \varphi(x_r) \right\rangle\!\right\rangle = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\text{pairs}} \frac{1}{i} \Delta(x_{i_1} - x_{i_2}) \ldots \frac{1}{i} \Delta(x_{i_{r-1}} - x_{i_r}) & \text{for } n \text{ even} \end{cases}$$

<u>difference</u> to previously considered case:
in addition to the product $\varphi(x_1) \ldots \varphi(x_n)$ with <u>different</u> space-time points $x_1, \ldots, x_n$ also terms from the interaction $\sim \varphi(y)^4$ with fields at the <u>same</u> space-time point y

pairing rule (Wick's theorem) $\rightarrow$ form all possible pairings of the $n + 4k$ fields and replace the various pairs by $\frac{1}{i} \Delta(x_p - x_q)$

<u>two</u> <u>types</u> of points:

<u>vertices</u> $y_1, \dots, y_k$

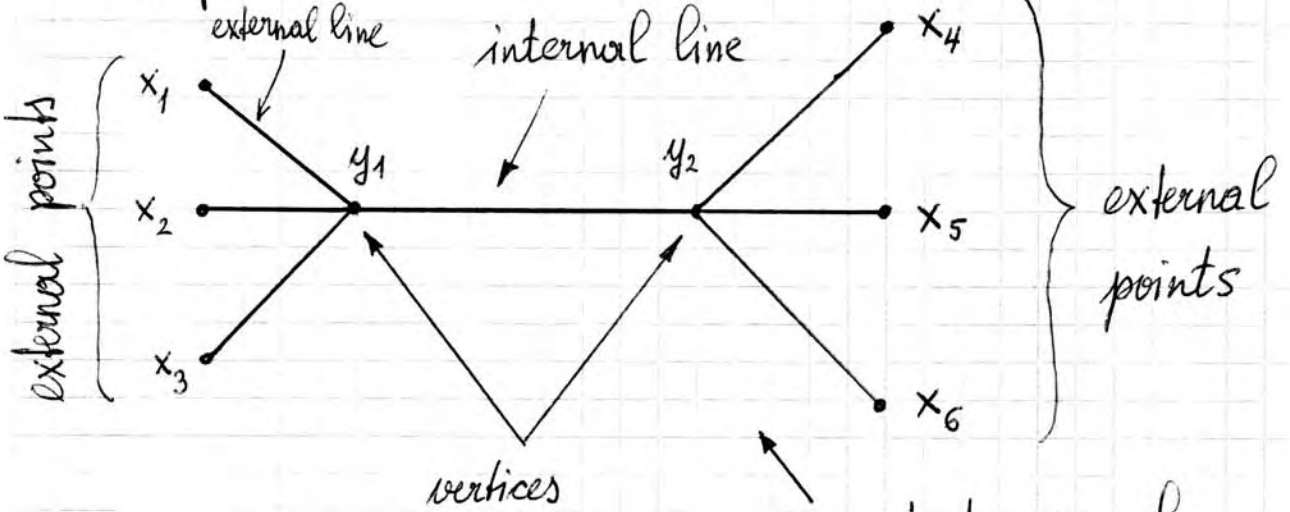
<u>external points</u> $x_1, \dots, x_n$

<u>two</u> <u>types</u> of lines connecting the points:

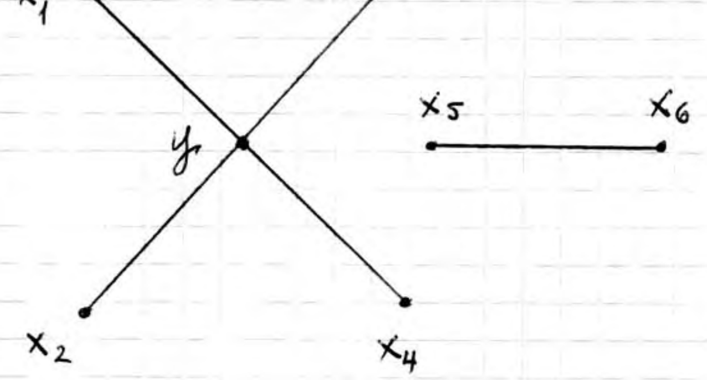
<u>internal lines</u>: connect vertices

<u>external lines</u>: at least one endpoint is external point

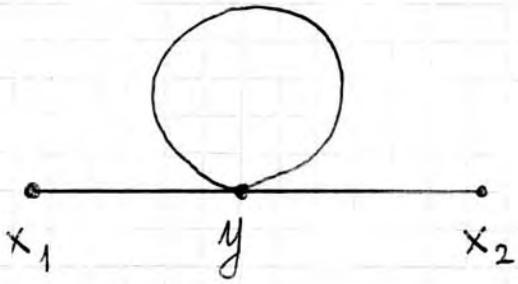
<u>examples</u>:



<u>connected</u> graph



<u>disconnected</u> graph



one-loop graph $\Delta(y-y) = \Delta(0)$

$\Delta(x)$ singular for $x^2 = 0$

UV divergence for $d \to 4$

UV divergences $\longrightarrow$ UV-regularization necessary

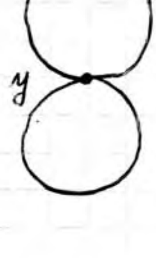
brute force method: stay in $d=4$, $\int d^4k$ made finite by cut-off $|k^\mu| < \Lambda$ $\to$ violates Lorentz invariance (in intermediate steps of the calculation) $\to$ Lorentz invariance only recovered in final results

dimensional regularization respects Lorentz invariance (and other symmetries of the theory under consideration)

renormalization $\to$ finite result for observables in the limit $d \to 4$ (or $\Lambda \to \infty$)

IR divergences generated by $\int d^d y \ldots$ in (disconnected) vacuum bubbles

example:



$\sim \int d^d y \; \Delta(0)\Delta(0)$

$\sim VT$ (volume of space-time)

contributions of disconnected vacuum bubbles <u>cancel</u> in the ratio

$$\frac{\left\langle\!\left\langle e^{iS_{int}} \varphi(x_1) \dots \varphi(x_n) \right\rangle\!\right\rangle}{\left\langle\!\left\langle e^{iS_{int}} \right\rangle\!\right\rangle}$$

→ IR divergences do not cause any problems (situation a bit more complicated in massless theories)

<u>n-point functions in φ^4-theory:</u>

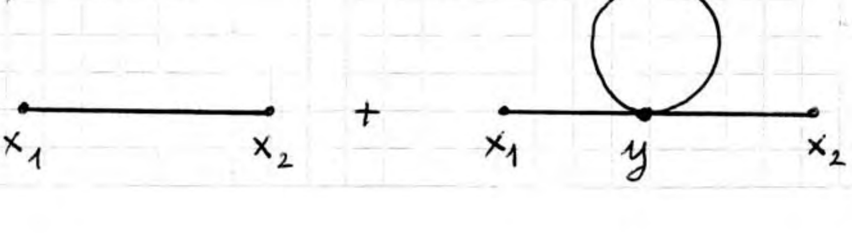
non-vanishing result only for <u>even</u> n (reason: symmetry $\varphi \to -\varphi$ of φ^4-theory)

two-point function to $\mathcal{O}(\lambda)$:

$$\left\langle\!\left\langle e^{iS_{int}} \varphi(x_1)\varphi(x_2) \right\rangle\!\right\rangle'$$ ← without disconnected vacuum bubbles

$$= \left\langle\!\left\langle \varphi(x_1)\varphi(x_2) \right\rangle\!\right\rangle - \frac{i\lambda}{4!}\int d^4y \left\langle\!\left\langle \varphi(x_1)\varphi(x_2)\varphi(y)^4 \right\rangle\!\right\rangle'$$

$+ \mathcal{O}(\lambda^2)$



$+ \mathcal{O}(\lambda^2)$

$$\langle\!\langle \varphi(x_1)\,\varphi(x_2)\,\varphi(y)^4 \rangle\!\rangle' =$$

$$= 4\,\frac{1}{i}\Delta(x_1-y)\;3\,\frac{1}{i}\Delta(x_2-y)\;\frac{1}{i}\Delta(0)$$

$$\langle 0|T\phi(x_1)\,\phi(x_2)|0\rangle = \frac{1}{i}\Delta(x_1-x_2)$$

$$+\frac{\lambda}{2}\,\Delta(0)\int d^dy\;\Delta(x_1-y)\,\Delta(x_2-y)\;+O(\lambda^2)$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2-k^2-i\varepsilon}$$

$$+\frac{\lambda}{2}\,\Delta(0)\int d^4y\int\frac{d^dk_1}{(2\pi)^d}\,\frac{e^{-ik_1(x_1-y)}}{m^2-k_1^2-i\varepsilon}\int\frac{d^dk_2}{(2\pi)^d}\,\frac{e^{-ik_2(x_2-y)}}{m^2-k_2^2-i\varepsilon}+O(\lambda^2)$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2-k^2-i\varepsilon}$$

$$+\frac{\lambda}{2}\,\Delta(0)\int\frac{d^dk_1}{(2\pi)^d}\,\frac{d^dk_2}{(2\pi)^d}\,\frac{e^{-ik_1x_1}\,e^{-ik_2x_2}}{(m^2-k_1^2-i\varepsilon)(m^2-k_2^2-i\varepsilon)}\,(2\pi)^d\,\delta^{(d)}(k_1+k_2)$$

$$+\,O(\lambda^2)$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2-k^2-i\varepsilon}+\frac{\lambda\Delta(0)}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{(m^2-k^2-i\varepsilon)^2}$$

$$+\,O(\lambda^2)$$

$$\langle 0|T\phi(x_1)\,\phi(x_2)\,|0\rangle =$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2-k^2-i\varepsilon}\left(1+\frac{i\lambda\Delta(0)}{2}\,\frac{1}{m^2-k^2-i\varepsilon}\right)+O(\lambda^2)$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2-k^2-i\varepsilon}\,\frac{1}{1-\frac{i\lambda\Delta(0)}{2}\,\frac{1}{m^2-k^2-i\varepsilon}}+O(\lambda^2)$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2-\frac{i\lambda\Delta(0)}{2}-k^2-i\varepsilon}+O(\lambda^2)$$

- general structure of two-point function (arbitrary order in perturbative expansion):

$$\langle 0|T\phi(x_1)\,\phi(x_2)\,|0\rangle =$$

$$= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{-ik(x_1-x_2)}}{m^2+\Sigma(k^2)-k^2-i\varepsilon}$$

remark: one-loop contribution to $\Sigma(k^2)$ in φ^4-theory independent of k^2; k^2-dependence arises only at two loops in the case of φ^4-theory pole of the propagator (in momentum space) <u>shifted</u> by higher order corrections

<u>physical interpretation:</u>

$$\langle 0 | T \phi(x) \phi(0) | 0 \rangle =$$

$$= \Theta(x^0) \langle 0 | \phi(x) \phi(0) | 0 \rangle$$

$$+ \Theta(-x^0) \langle 0 | \phi(0) \phi(x) | 0 \rangle$$

insert complete ONS of momentum eigenstates

$$= \Theta(x^0) \sum_\alpha \langle 0 | \phi(x) | \alpha \rangle \langle \alpha | \phi(0) | 0 \rangle$$

$$+ \Theta(-x^0) \sum_\alpha \langle 0 | \phi(0) | \alpha \rangle \langle \alpha | \phi(x) | 0 \rangle$$

$$= \Theta(x^0) \sum_\alpha \langle 0 | e^{iPx} \phi(0) e^{-iPx} | \alpha \rangle \langle \alpha | \phi(0) | 0 \rangle$$

$$+ \Theta(-x^0) \sum_\alpha \langle 0 | \phi(0) | \alpha \rangle \langle \alpha | e^{iPx} \phi(0) e^{-iPx} | 0 \rangle$$

$$= \Theta(x^0) \sum_\alpha e^{-ip_\alpha x} | \langle 0 | \phi(0) | \alpha \rangle |^2$$

$$+ \Theta(-x^0) \sum_\alpha e^{ip_\alpha x} | \langle 0 | \phi(0) | \alpha \rangle |^2$$

$p^0 = \sqrt{m_{ph}^2 + \vec{p}^2}$

$$\sum_\alpha |\alpha\rangle\langle\alpha| = |0\rangle\langle 0| + \int d\mu(p)\, |p\rangle\langle p|$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \int d\mu(p_1) \ldots d\mu(p_n)\, |p_1, \ldots, p_n; {}^{in}_{out}\rangle\langle p_1, \ldots, p_n; {}^{in}_{out}|$$

in our case no contribution from vacuum state as $\langle 0|\phi(0)|0\rangle = 0$ (in theories with $\langle 0|\phi(0)|0\rangle \neq 0$ take $\phi(x) - \langle 0|\phi(0)|0\rangle$ instead of $\phi(x)$)

$$\Rightarrow \quad \langle 0|T\phi(x)\,\phi(0)|0\rangle =$$

$$= \Theta(x^0) \int d\mu(p)\, e^{-ipx}\, |\langle 0|\phi(0)|p\rangle|^2$$

$$+ \Theta(-x^0) \int d\mu(p)\, e^{+ipx}\, |\langle 0|\phi(0)|p\rangle|^2$$

+ contributions from intermediate states with $n \geq 2$

$\langle 0|\phi(0)|p\rangle$ is <u>independent</u> of p because of Lorentz invariance:

$|p\rangle \equiv |\sqrt{m_{ph}^2+\vec{p}^2}, \vec{p}\rangle$ can be obtained by a Lorentz transformation acting on $|m_{ph}, \vec{0}\rangle$:

$$|p\rangle = |\sqrt{m_{ph}{}^2+\vec{p}^2}, \vec{p}\rangle = U(L)\,|m_{ph}, \vec{0}\rangle,$$

where $L\,(m_{ph}, \vec{0}) = (\sqrt{m_{ph}^2+\vec{p}^2}, \vec{p})$

$$\Rightarrow \langle 0|\phi(0)|p\rangle = \langle 0|\phi(0)\,U(L)|m_{ph}, \vec{0}\rangle$$

$$\underset{\substack{\uparrow \\ U(L)|0\rangle = |0\rangle}}{=} \langle 0|U(L)^{-1}\phi(0)\,U(L)\,|m_{ph}, \vec{0}\rangle =$$

$$\underset{\substack{\uparrow \\ \phi \text{ is a scalar field}}}{=} \langle 0|\phi(0)\,|m_{ph}, \vec{0}\rangle \qquad \underline{\text{independent}} \text{ of } p$$

$$U(L)^{-1}\phi(x)\,U(L) = \phi\,(L^{-1}x)$$

$$Z := |\langle 0|\phi(0)|p\rangle|^2$$

with a suitable definition of the phase of $|p\rangle$:

$$\sqrt{Z} = \langle 0|\phi(0)|p\rangle$$

$$\Rightarrow \quad \langle 0 | T \phi(x) \phi(0) | 0 \rangle =$$

$$= Z \underbrace{\left\{ \Theta(x^0) \int d\mu(p)\, e^{-ipx} + \Theta(-x^0) \int d\mu(p)\, e^{ipx} \right\}}_{\frac{1}{i} \Delta(x; m_{ph}^2)} + \ldots$$

$$= \frac{Z}{i} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ikx}}{m_{ph}^2 - k^2 - i\varepsilon} + \ldots$$

pole of the two-point function in momentum space determines the <u>physical</u> mass m_{ph}

$$\Rightarrow \quad m^2 + \Sigma(m_{ph}^2) - m_{ph}^2 = 0$$

residue of the pole at $k^2 = m_{ph}^2$ occuring in

$$\frac{1}{m^2 + \Sigma(k^2) - k^2 - i\varepsilon}$$

may be worked out by expanding the function $\Sigma(k^2)$ around this point:

$$\Sigma(k^2) = \Sigma(m_{ph}^2) + (k^2 - m_{ph}^2)\Sigma'(m_{ph}^2) + O[(k^2 - m_{ph}^2)^2] \Rightarrow Z = \frac{1}{1 - \Sigma'(m_{ph}^2)}$$