

3. Functional integral (path integral)

quantum mechanics:

$$\langle 0 | T Q(t_1) \dots Q(t_n) | 0 \rangle = \frac{1}{N} \int [dq] e^{\frac{i}{\hbar} S[q]} q(t_1) \dots q(t_n)$$

$\uparrow$ operators $\uparrow$ ground state $\uparrow$ c-numbers

$$S[q] = \int_{-\infty}^{+\infty} dt \left[\frac{1}{2} \dot{q}(t)^2 - V(q(t)) \right] \text{ classical action}$$

$$N = \int [dq] e^{\frac{i}{\hbar} S[q]} \quad \text{normalization constant}$$

$[dq]$ integration measure (classical theory $\rightarrow$ quantum theory)

metric in the space of all paths $\rightarrow$ volume element

consider two neighbouring paths $q(t)$, $q(t) + dq(t)$
 (distance)² between two paths:

$$ds^2 = \int dt [dq(t)]^2$$

$$q(t) = \sum_{i=1}^{\infty} q_i u_i(t)$$

$\{u_i\}_{i=1}^{\infty}$ complete set of orthonormal functions

$$\int dt u_i(t) u_j(t) = \delta_{ij}$$

regularization: restrict the space of paths to those for which only the first N components $q_1, \dots, q_N$ are different from zero:

$$q(t) = q_1 u_1(t) + \dots + q_N u_N(t)$$

$$ds^2 = (dq_1)^2 + \dots + (dq_N)^2$$

$$\Rightarrow [dq]_{\text{reg}} = dq_1 \dots dq_N$$

$$\int [dq] \dots = \lim_{N \rightarrow \infty} \int dq_1 \dots dq_N \dots$$

important property of the measure: translation invariance

$$q(t) = q'(t) + R(t) \quad (\text{arbitrary function } R(t))$$

$\rightarrow$ metric unchanged $\Rightarrow$ volume element induced by

$$\text{metric unchanged} \Rightarrow [dq] = [dq']$$

explicit construction of measure in terms of a metric

not indispensable $\rightarrow$ translation invariance sufficient

relation between classical mechanics and quantum theory:

rapid oscillation of $e^{\frac{i}{\hbar} S[q]}$ when $q \rightarrow q + \delta q$

in the classical limit $\hbar \rightarrow 0$, except in the neighbourhood of the path where the phase is stationary:

$$\left. \frac{\delta S[q]}{\delta q(t)} \right|_{q=q_{cl}} = 0$$



$m \ddot{q}_{cl}(t) = -V'(q_{cl})$ classical equation of motion

→ significant contribution to the path integral only from this region around $q_{cl}(t)$

→ laws of classical mechanics can be formulated as action principle

region which dominates the path integral for $\hbar \rightarrow 0$

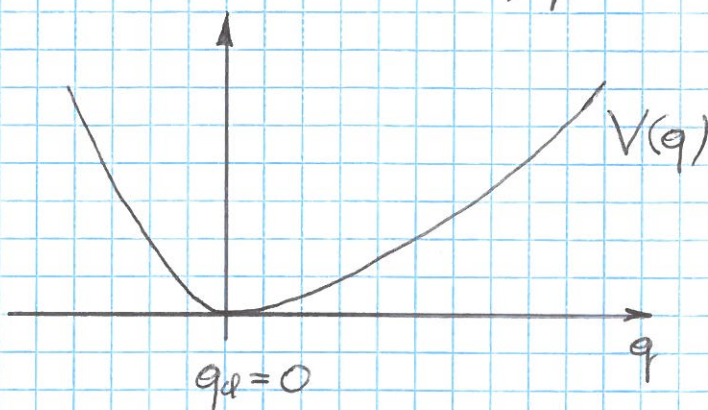
characterized by $|S[q] - S[q_{cl}]| \lesssim \hbar$: only there the contributions interfere constructively

$$q(t) = q_{cl}(t) + \delta q_{cl}(t)$$

$$S[q] - S[q_{cl}] \sim (\delta q)^2 \quad (\text{no linear term})$$

→ path integral dominated by those paths for which $q(t) = q_{cl}(t) + \mathcal{O}(\sqrt{\hbar})$

relevant classical solution for vacuum expectation values:
classical groundstate characterized by $V'(q_{cl}) = 0$;
choose coordinates such that minimum of potential occurs at $q_{cl} = 0$:



$\Rightarrow q(t) \sim \sqrt{\hbar}$ generate significant contributions to the path integral

$$\Rightarrow \langle 0 | T Q(t_1) Q(t_2) | 0 \rangle = \frac{1}{N} \int [dq] e^{\frac{i}{\hbar} S[q]} q(t_1) q(t_2) \sim \mathcal{O}(\hbar)$$

exercises: calculate $\langle 0 | T Q(t_1) Q(t_2) | 0 \rangle$ for the harmonic oscillator using both the operator approach and the path integral method

quantum field theory:

$$\langle 0 | \underset{\substack{\uparrow \\ \text{operators}}}{T} \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{1}{N} \int [d\phi] e^{iS[\phi]} \underset{\substack{\uparrow \\ \text{c-number fields}}}{\phi(x_1) \dots \phi(x_n)}$$

remark: back to $\hbar = 1$

$$N = \int [d\phi] e^{iS[\phi]}$$

$S[\phi]$... classical action of the model, i.e.

$$S[\phi] = \int d^4x \frac{1}{2} [\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2]$$

(free scalar field)

integration measure $[d\phi]$: immediate generalization of the one in quantum mechanics

distance between two neighbouring field configurations $\phi(x)$, $\phi(x) + d\phi(x)$ given by

$$ds^2 = \int d^4x [d\phi(x)]^2$$

→ $[d\phi]$ is volume element induced by this metric

shift $\varphi(x) = \varphi'(x) + k(x)$ leaves distance unchanged

$$\Rightarrow [d\varphi] = [d\varphi']$$

generating functional

$$Z[f] = \frac{1}{N} \int [d\varphi] e^{iS[\varphi]} e^{i \int d^4x f(x) \varphi(x)}$$

↑
external field (source)

$$= \frac{1}{N} \int [d\varphi] e^{iS[\varphi]} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 f(x_1) \varphi(x_1) \dots \int d^4x_n f(x_n) \varphi(x_n)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n f(x_1) \dots f(x_n) \frac{1}{N} \int [d\varphi] e^{iS[\varphi]} \varphi(x_1) \dots \varphi(x_n)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n f(x_1) \dots f(x_n) \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

$$=: \langle 0 | T e^{i \int d^4x f(x) \phi(x)} | 0 \rangle$$

n -point functions can be obtained as functional derivative of $Z[f]$ with respect to the external field, taken at $f=0$:

$$\langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{1}{i^n} \frac{\delta^n Z[f]}{\delta f(x_1) \dots \delta f(x_n)} \Big|_{f=0}$$

remark: $\frac{\delta f(x)}{\delta f(y)} = \delta^{(4)}(x-y)$; $Z[0] = 1$

computation of $Z[f]$ for the free scalar field:

$$Z[f] = \frac{1}{N} \int [d\varphi] e^{i \int d^4x \left[\underbrace{\frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{m^2}{2} \varphi(x)^2}_{-\frac{1}{2} \varphi(x) \square \varphi(x)} + f(x) \varphi(x) \right]}$$

(partial integration)

Gaussian functional integral with purely imaginary exponent (Fresnel type integral) $\rightarrow$ introduce damping factor by $m^2 \rightarrow m^2 - i\varepsilon$ ($\varepsilon > 0$); $\varepsilon \rightarrow 0$ after integration

shift of integration variable: $\varphi(x) = \varphi'(x) + k(x)$

$\nearrow$
new integration variable

choose $k(x)$ such that terms linear in $\varphi'(x)$ disappear:

$$\begin{aligned} & \int d^4x \left[-\frac{1}{2} \varphi(x) (\square + m^2) \varphi(x) + f(x) \varphi(x) \right] = \\ & = \int d^4x \left[-\frac{1}{2} (\varphi'(x) + k(x)) (\square + m^2) (\varphi'(x) + k(x)) \right. \\ & \quad \left. + f(x) (\varphi'(x) + k(x)) \right] = \end{aligned}$$

$$= \int d^4x \underbrace{\left[-\frac{1}{2} \varphi'(x) (\square + m^2 - i\varepsilon) \varphi'(x) \right]}_{\text{quadratic in } \varphi'(x)}$$

$$\underbrace{-\frac{1}{2} \varphi'(x) (\square + m^2 - i\varepsilon) \kappa(x) - \frac{1}{2} \kappa(x) (\square + m^2 - i\varepsilon) \varphi'(x) + f(x) \varphi'(x)}_{\text{linear in } \varphi'(x)}$$

$$\underbrace{-\frac{1}{2} \kappa(x) (\square + m^2 - i\varepsilon) \kappa(x) + f(x) \kappa(x)}_{\text{independent of } \varphi'(x)}$$

$$\Rightarrow \kappa(x) \text{ must fulfil } (\square + m^2 - i\varepsilon) \kappa(x) = f(x)$$

$$\Rightarrow Z[f] = \frac{1}{N} \int [d\varphi'] e^{-\frac{i}{2} \int d^4x \varphi'(x) (\square + m^2 - i\varepsilon) \varphi'(x)} \\ \times e^{\frac{i}{2} \int d^4x f(x) \kappa(x)}$$

$$\text{but } N = \int [d\varphi] e^{-\frac{i}{2} \int d^4x \varphi(x) (\square + m^2 - i\varepsilon) \varphi(x)}$$

$$\Rightarrow Z[f] = e^{\frac{i}{2} \int d^4x f(x) \kappa(x)}$$

$$\rightarrow \text{have to find solution of } (\square + m^2 - i\varepsilon) \kappa(x) = f(x)$$

the corresponding homogeneous equation does not admit physically meaningful solutions :

$(p^0)^2 - \vec{p}^2 = m^2 - i\varepsilon$ implies that for real three-momentum, the frequency contains a small imaginary part

$\rightarrow e^{\pm i p x}$ explodes either for $x^0 \rightarrow +\infty$ or $x^0 \rightarrow -\infty$

$\Rightarrow K(x)$ is uniquely determined by $(\square + m^2 - i\varepsilon)K(x) = f(x)$

we know already : $(\square + m^2 - i\varepsilon)\Delta(x) = \delta^{(4)}(x)$

$$\Rightarrow K(x) = \int d^4y \Delta(x-y) f(y)$$

$$\begin{aligned} \Rightarrow Z[f] &= \langle 0 | T e^{i \int d^4x f(x) \phi(x)} | 0 \rangle \\ &= e^{\frac{i}{2} \int d^4x d^4y f(x) f(y) \Delta(x-y)} \end{aligned}$$

expansion of the r.h.s. in powers of f contains

only even terms $\Rightarrow \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = 0$

if n is odd (e.g. $\langle 0 | \phi(x) | 0 \rangle = 0$)

two-point function: $\frac{i^2}{2!} \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \frac{i}{2} \Delta(x-y)$

$$\Rightarrow \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \frac{1}{i} \Delta(x-y)$$

for n even:

$$\int d^4x_1 \dots d^4x_n f(x_1) \dots f(x_n) \frac{i^n}{n!} \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

$$= \int d^4x_1 \dots d^4x_n f(x_1) \dots f(x_n) \frac{i^p}{2^p p!} \Delta(x_1 - x_2) \dots \Delta(x_{n-1} - x_n)$$

$$p = \frac{n}{2}$$

integrand on the l.h.s. symmetric with respect to an interchange of the arguments, but integrand on the r.h.s. is not

only totally symmetric parts of the two integrands are the same!

$\Delta(-x) = \Delta(x) \Rightarrow$ it suffices to consider all possible partitions of the set $1, \dots, n$ into pairs

$$(i_1, i_2), \dots, (i_{n-1}, i_n) \rightarrow (n-1)(n-3) \dots 1 = (n-1)!!$$

such pairings $\rightarrow$ use $2^p p! (2p-1)!! = (2p)!$

$$\frac{i^{2p}}{(2p)!} \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{i^p}{2^p p!} \frac{2^p p!}{(2p)!} \sum_{\text{pairings}} \Delta(x_{i_1} - x_{i_2}) \dots$$

$$\dots \Delta(x_{i_{n-1}} - x_{i_n})$$

$$\Rightarrow \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \sum_{\text{pairings}} \frac{1}{i} \Delta(x_{i_1} - x_{i_2}) \dots \frac{1}{i} \Delta(x_{i_{n-1}} - x_{i_n})$$

graphic representation :

- (i) represent the arguments $(x_1, \dots, x_n)$ of the n -point function by points in the plane (non-vanishing Green function only for even n)
- (ii) draw the graphs that arise by decomposing the set of points into pairs indicating this with a line between the two points
- (iii) for each line connecting the points x, y denote a factor $\frac{1}{i} \Delta(x-y)$; the contribution associated with a given graph is the product of the factors coming from the various lines
- (iv) final result is the sum of the contributions from the different graphs

example: four-point function of a free scalar field



$$\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle =$$

$$= \frac{1}{i} \Delta(x_1 - x_2) \frac{1}{i} \Delta(x_3 - x_4)$$

$$+ \frac{1}{i} \Delta(x_1 - x_3) \frac{1}{i} \Delta(x_2 - x_4)$$

$$+ \frac{1}{i} \Delta(x_1 - x_4) \frac{1}{i} \Delta(x_2 - x_3)$$

$(4-1)!! = 3$ different pairings $\rightarrow 3$ graphs