1. Introduction

consequences of relativity + quantum theory:

at sufficiently high energies, particles can be created and destroyed

examples (for initial state pp):

$pp \rightarrow pp\pi^0$ possible once total energy of incoming particles > threshold energy $(2m_p + m_\pi)c^2$;

at higher energies $pp \rightarrow ppp\bar{p}$ can occur

$(E_{thr} = 4m_p c^2)$

remark: $pp \rightarrow pp$ "elastic scattering"

further examples (initial state e^+e^-):

$e^+e^- \rightarrow e^+e^-$ (elastic electron-positron scattering)

$e^+e^- \rightarrow e^+e^-\gamma$ ($E_{thr} = 2m_e c^2$; $m_\gamma = 0 \rightarrow$ photons can be "soft")

$e^+e^- \rightarrow e^+e^-\gamma\gamma, \ldots$ —//—

$e^+e^- \rightarrow \gamma\gamma$ (pair annihilation)

$e^+e^- \rightarrow \pi^+\pi^-$ (pion production)

$e^+e^- \rightarrow 2e^+ + 2e^-$ ($E_{thr} = 4m_e c^2$)

→ appropriate theory must be a <u>relativistic</u> <u>many-particle</u> theory (in general $\underline{\Delta N \neq 0}$)

→ <u>relativistic quantum field theory</u>

<u>in contrast</u>: non-relativistic quantum mechanics (NRQM) applicable only for processes with $\underline{v \ll c}$ and $\underline{\Delta N = 0}$

T2: essentialy <u>one-particle</u> NRQM

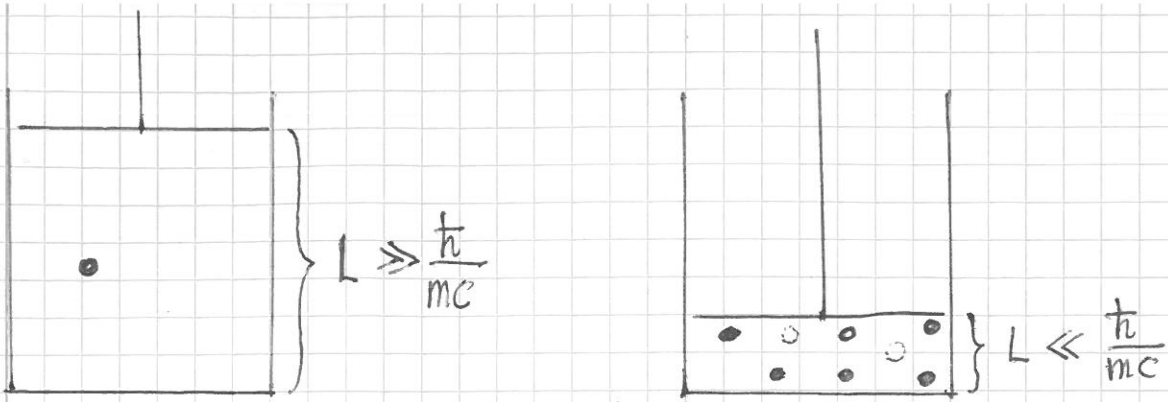
NRQM assumes <u>localizability</u> of one single particle down to <u>arbitrarily small distances</u> (positition eigen-vectors $|\vec{x}\rangle$, position operator $\vec{X}$ with $\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$, wave function $\psi(\vec{x}) = \langle \vec{x} | \psi \rangle$)

uncertainty relation $\Delta x \, \Delta p \geq \hbar/2$: $\Delta X \to 0$ implies <u>only</u> $\Delta p \to \infty$

<u>however</u>: once relativistic energies (momenta) are reached in this localization process <u>further</u> particles can be produced → concept of one-particle description <u>breaks down</u>!

relevent order of magnitude : $\Delta p \sim mc$

$\Rightarrow \Delta x \sim \frac{\hbar}{mc} = \lambda_c$ (Compton length of the particle)



→ particle with mass m <u>cannot</u> be localized in a region with $L \lesssim \lambda_c = \frac{\hbar}{mc}$

at smaller distances more particles are produced

→ particle number in container <u>undetermined</u>

<u>numerical example</u>: $m_e = 0.511\ \mathrm{MeV}/c^2$

$$\lambda_e = \frac{\hbar}{m_e c} = \frac{1}{m_e c^2} \underbrace{\hbar c}_{197\ \mathrm{MeV} \times \mathrm{fm}} = \frac{197\ \mathrm{MeV} \times \mathrm{fm}}{0.511\ \mathrm{MeV}} \simeq 4 \times 10^{-13}\,\mathrm{m}$$

relevant length scale in atomic physics:

Bohr radius $a \simeq 5 \times 10^{-11}\,\mathrm{m} \gg \lambda_e$ → NRQM applicable in atomic physics → <u>no</u> problem with localizability of e^- in atoms, relativistic correction to kinetic energy <u>small</u> ($v \sim \alpha c$!), correction due to many-particle (intermediate) states also <u>small</u>

<u>remark</u>: relativistic corrections in atomic physics

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2 + \frac{mv^2}{2} \underbrace{- \frac{3}{4} mv^2 \frac{v^2}{c^2}} + \dots$$

suppressed by factor $\sim \frac{v^2}{c^2}$ relative to $\frac{mv^2}{2}$

contributions from multiparticle (intermediate) states to energy levels of atoms:

$$H = \underbrace{H_0}_{\text{nonrelativistic part}} + H' \qquad H_0 |m\rangle = E_m^{(0)} |m\rangle$$

$$\delta E_m = \langle m | H' | m \rangle + \sum_{n \neq m} \frac{|\langle m | H' | n \rangle|^2}{E_m - E_n} + \dots$$

also contributions from multiparticle states

→ suppressed by factor $\frac{E}{mc^2} \sim \frac{mv^2}{mc^2} = \left(\frac{v}{c}\right)^2$

~typical atomic energy

typical energy denominator

→ corrections by relativistic kinematics and multiparticle states (in general) of same order

<u>important conclusion</u>: $\nexists$ consistent relativistic <u>one-particle</u> quantum mechanics!

$\nexists$ position operator $\vec{X}$ in a relativistic theory

$\nexists$ basis $\{|\vec{x}\rangle\}$ of position eigenstates —"—

<u>reason</u>: a particle <u>cannot</u> be localized in an arbitrarily small spatial region!

→ it <u>does not</u> make sense to construct a relativistic one-particle quantum mechanical "theory" by just replacing the nonrelatic Hamiltonian $\vec{P}^2/2m$ by the relativistic expression $H = c\sqrt{\vec{P}^2 + m^2c^2}$ but still using position eigenstates $|\vec{x}\rangle$! → it is instructive to see at which point such a naive "theory" finally fails: to this end, we compute the evolution kernel $\langle \vec{x} | e^{-iHt/\hbar} | \vec{0} \rangle$, which should give us the amplitude to find the particle at the point $\vec{x}$ at time t if it had been at the origin at time 0 → it turns out that this evolution kernel <u>violates causality</u>, as it <u>does not</u> vanish outside the light cone!

$$\langle \vec{x} | e^{-iHt/\hbar} | \vec{0} \rangle = \int d^3p \, \langle \vec{x} | e^{-iHt/\hbar} | \vec{p} \rangle \langle \vec{p} | \vec{0} \rangle$$

$$= \int \frac{d^3p}{(2\pi\hbar)^3} \, e^{i\vec{p}\cdot\vec{x}/\hbar} \, e^{-ic\sqrt{\vec{p}^2 + m^2c^2}\, t/\hbar}$$

$r = |\vec{x}|$

$$= \int_0^\infty \frac{dp\, p^2}{(2\pi\hbar)^3} \int_{\cos\theta=-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi \; e^{ipr\cos\theta/\hbar} \, e^{-ic\sqrt{p^2+m^2c^2}\, t/\hbar}$$

$$= \frac{2\pi}{(2\pi\hbar)^3} \int_0^\infty dp\, p^2 \, e^{-ic\sqrt{p^2+m^2c^2}\, t/\hbar} \; \frac{e^{ipr/\hbar} - e^{-ipr/\hbar}}{ipr/\hbar}$$

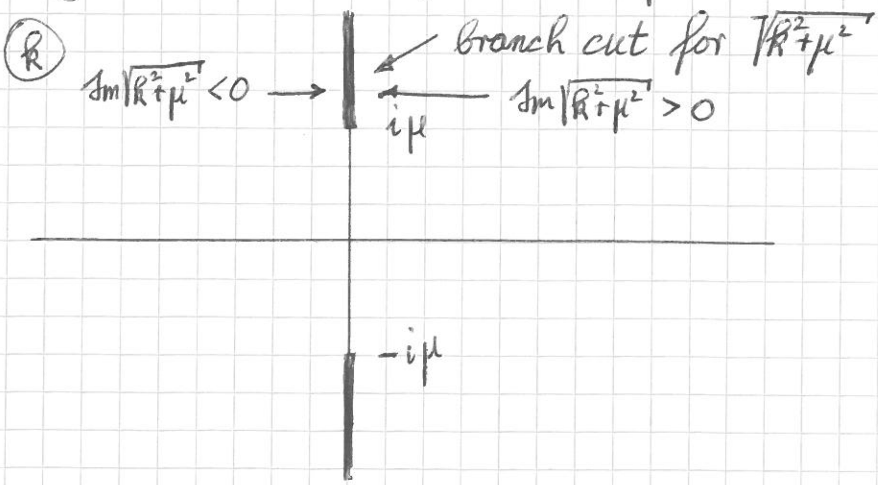
$$= -\frac{i}{(2\pi\hbar)^2 r} \int_0^\infty dp\, p \left(e^{ipr/\hbar} - e^{-ipr/\hbar} \right) e^{-ic\sqrt{p^2+(mc)^2}\, t/\hbar}$$

$p = \hbar k$

$$= -\frac{i}{(2\pi)^2 r} \int_{-\infty}^{+\infty} dk\, k \, e^{ikr} \, e^{-ic\sqrt{k^2+\mu^2}\, t}$$

$\mu = \frac{mc}{\hbar}$ inverse Compton length

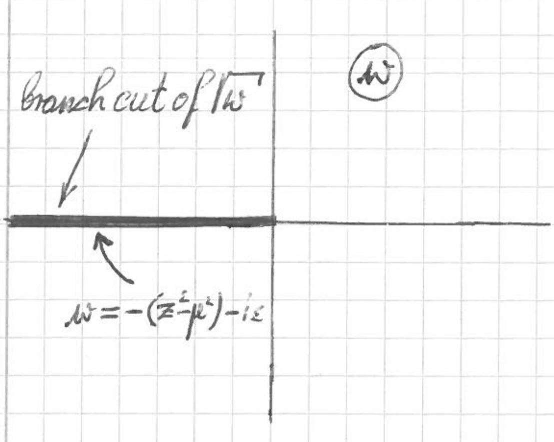
$\rightarrow$ integration in complex k-plane:



behaviour of $\sqrt{k^2+\mu^2}$ at left/right rim of upper branch cut:

<u>left</u>: $k = iz - \varepsilon$, $z > \mu$

$$\sqrt{k^2+\mu^2} = \sqrt{-z^2+\mu^2-i\varepsilon} = \sqrt{-\underbrace{(z^2-\mu^2)}_{>0}-i\varepsilon} = \sqrt{z^2-\mu^2}\,(-i)$$

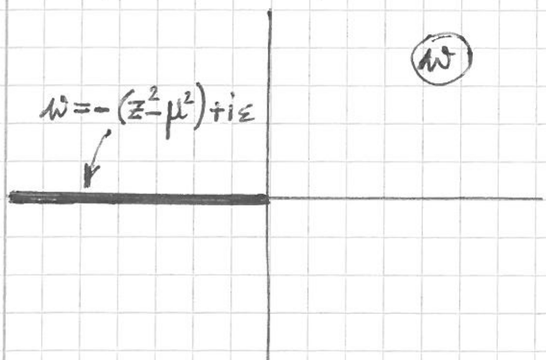


$w = |w|e^{i\theta} \Rightarrow \sqrt{w} = \sqrt{|w|}\,e^{i\theta/2}$

$-\pi < \theta < \pi$

<u>right</u>: $k = iz + \varepsilon$, $z > \mu$

$$\sqrt{k^2+\mu^2} = \sqrt{-z^2+\mu^2+i\varepsilon} = \sqrt{-(z^2-\mu^2)+i\varepsilon} = \sqrt{z^2-\mu^2}\;i$$



<u>left</u> rim of branch cut: $e^{-ict\sqrt{k^2+\mu^2}} = e^{-\underbrace{ct\sqrt{z^2-\mu^2}}_{>0}}$

decays exponentially for $z \to \infty$

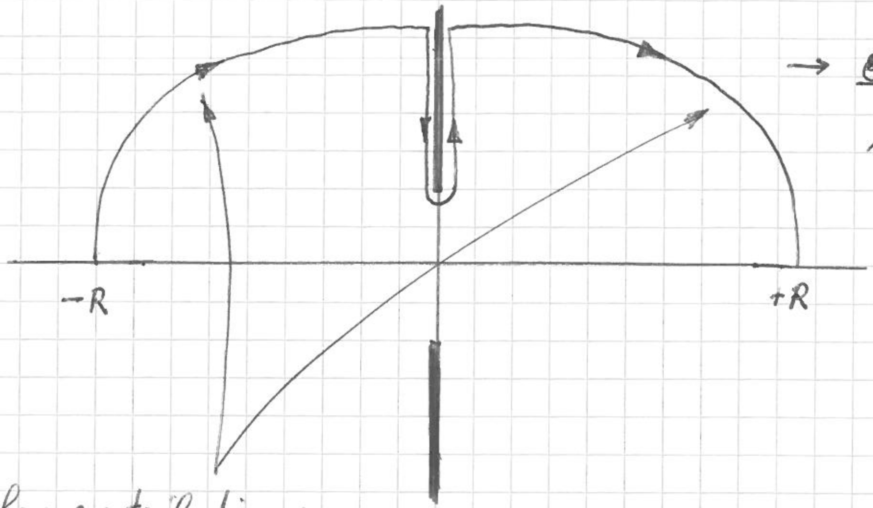
<u>right</u> rim of branch cut: $e^{-ict\sqrt{k^2+\mu^2}} = e^{+\underbrace{ct\sqrt{z^2-\mu^2}}_{>0}}$

grows exponentially for $z \to \infty$

e^{ikr} decays exponentially $(r > 0)$ for $k \to i\infty$ on both sides

<u>outside</u> of light cone $(r > ct)$:

product $e^{ikr} e^{-ict\sqrt{k^2+\mu^2}}$ decays always exponentially on <u>both</u> sides of the branch cut (e^{ikr} "wins" also on the right side because of $r > ct$)



→ <u>deformation</u> of path of integration

$$\Rightarrow \langle \vec{x} | e^{-iHt/\hbar} | \vec{0} \rangle = -\frac{i}{(2\pi)^2 r} \int_{-\infty}^{+\infty} dk\, k\, e^{ikr} e^{-ict\sqrt{k^2+\mu^2}}$$

$$= \frac{i}{(2\pi)^2 r} \int_{\mu}^{\infty} dz\, z\, e^{-zr} \left(e^{ct\sqrt{z^2-\mu^2}} - e^{-ct\sqrt{z^2-\mu^2}} \right)$$

contribution from right rim

contribution from left rim

$$= \frac{i}{2\pi^2 r} \int_{\mu}^{\infty} dz\, \underbrace{z\, e^{-zr} \sinh\left(ct\sqrt{z^2-\mu^2}\right)}_{>0} \neq 0$$

→ <u>violation</u> of <u>causality</u>!

this "theory" is obviously <u>sick</u> and leads to <u>contradictions:</u> we can measure a particle's position in this "theory" → we can trap it in a box of arbitrarily small size → we can release it and detect it outside the forward light cone (with a certain probability) → the particle can travel faster than light → the particle can move backwards in time, with all the associated paradoxes!

although probability that the particle is found outside its forward light cone falls off exponentially as one goes further from the light cone, it is still an <u>unacceptable contradiction</u>

<u>Lit.:</u> Sidney Coleman, Notes from Physics 253a, arXiv: 1110.5013.v5 [physics.ed-ph] 21 Feb 2013