Exercises "Particle Physics II" (2016)

Numerical values of physical constants and particle physics data can be found at pdg.lbl.gov (Particle Data Tables).

If not stated otherwise, we are using natural units ($\hbar = c = 1$).

1. The energy of an electron in the LEP collider (at maximum beam energy) was about 100 GeV. Compute the corresponding velocity of the electron and express your result in units of the speed of light.

2. The energy of a proton in the LHC (at maximum beam energy) will be 7 TeV. Compute the corresponding velocity of the proton and express your result in units of the speed of light.

3. A particle of mass M decays into two particles with masses m_1 and m_2. Express the momenta $|\vec{p}_{1,2}|$ and the energies $E_{1,2}$ of the two decay products in the rest frame of the decaying particle in terms of M, m_1 and m_2.

4. Apply the result of the previous problem to the decay $\pi^\pm \to \ell^\pm \overset{(-)}{\nu}_\ell$, where $\ell = e, \mu$. Determine the velocity of $e^\pm$ and $\mu^\pm$, respectively.

5. Determine the range of possible values of momentum and energy of the electron in the case of the β-decay of a free neutron ($n \to p\, e^- \bar{\nu}_e$). (The neutron is assumed to be at rest.)

6. Compute the Compton length of the pion. Discuss its physical relevance.

7. $\varphi(t, \vec{x})$ is a real (classical) field fulfilling the boundary conditions $\varphi(t_{1,2}, \vec{x}) = f_{1,2}(\vec{x})$ at some times $t_1 < t_2$. Assume further that $\varphi(t, \vec{x})$ minimizes the action integral
$$\int_{t_1}^{t_2} dt \int_{\mathbb{R}^3} d^3x \, \mathcal{L}\left(\varphi(t, \vec{x}), \partial_\mu \varphi(t, \vec{x})\right) \ ,$$
whith some Lagrangian density $\mathcal{L}$. Show that $\varphi(t, \vec{x})$ has to satisfy the Euler-Lagrange equation
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} = \frac{\partial \mathcal{L}}{\partial \varphi} \ .$$

8. Derive the field equation of φ^4 theory following from the Lagrangian density
$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 \ .$$

9. What is the energy-momentum tensor $T_{\mu\nu}$ of φ^4 theory? Verify $\partial^\mu T_{\mu\nu} = 0$.

10. The commutation relations of the creation and annihilation operators of a Hermitian scalar field (spin 0 field) with mass m are given by

$$\left[a(p), a(p')^\dagger\right] = \underbrace{(2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}')}_{\delta(p,p')} \ , \quad [a(p), a(p')] = 0 \ ,$$

where $p^0 = \sqrt{m^2 + \vec{p}^2}$. The one-particle momentum eigenstate $|p\rangle$ is defined by $|p\rangle = a(p)^\dagger|0\rangle$. The general form of a normalizable one-particle state $|\psi^{(1)}\rangle$ is given by

$$|\psi^{(1)}\rangle = \int \underbrace{\frac{d^3p}{(2\pi)^3 2p^0}}_{d\mu(p)} |p\rangle\, \psi^{(1)}(p) \ .$$

(a) $\langle p|p'\rangle = ?$

(b) Determine the normalization condition for the momentum-space wave function $\psi^{(1)}(p)$ implied by the state normalization $\langle\psi^{(1)}|\psi^{(1)}\rangle = 1$.

(c) Show that the projection operator

$$P^{(1)} = \int d\mu(p)|p\rangle\langle p|$$

satisfies indeed $P^{(1)}P^{(1)} = P^{(1)}$.

11. The two-particle momentum eigenstate $|p_1, p_2\rangle$ is defined by

$$|p_1, p_2\rangle = a(p_1)^\dagger a(p_2)^\dagger |0\rangle \ .$$

(a) $\langle p_1, p_2|p_1' p_2'\rangle = ?$

(b) Determine the operator $P^{(2)}$ projecting on the two-particle subspace.

(c) Discuss the general form of a normalizable two-particle state $|\psi^{(2)}\rangle$ and the properties of the corresponding two-particle wave function in momentum space.

12. In the case of n particles, one defines

$$|p_1, \ldots p_n\rangle = a(p_1)^\dagger \ldots a(p_n)^\dagger |0\rangle \ .$$

Show by induction:

$$\langle p_1, \ldots p_n|k_1, \ldots k_n\rangle = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \delta(p_i, k_{\sigma(i)}) \ .$$

13. Discuss the projection operator $P^{(n)}$ and the general form of an n-particle state $\psi^{(n)}$.

14. The Fourier decomposition of a real scalar field is given by

$$\phi(x) = \int d\mu(p) \left[a(p) e^{-ipx} + a(p)^\dagger e^{ipx} \right] .$$

Show that $a(p)$ and $a(p)^\dagger$ can be obtained from $\phi(x)$ by the relations

$$a(p) = i \int d^3x \, e^{ipx} \overleftrightarrow{\partial_0} \, \phi(x) \, , \quad a(p)^\dagger = -i \int d^3x \, e^{-ipx} \overleftrightarrow{\partial_0} \, \phi(x) \, .$$

15. Use the previous formulas to show that the canonical equal-time commutation relations for ϕ and $\dot{\phi}$ imply the commutation relations for $a(p)$ and $a(p)^\dagger$ displayed in problem 1.

16. The four-momentum operator P^μ is given by

$$P^\mu = \int d\mu(p) \, p^\mu \, a(p)^\dagger a(p) \, .$$

Show the following commutation relations:

$$[P^\mu, a(p)] = -p^\mu a(p), \quad \left[P^\mu, a(p)^\dagger\right] = p^\mu a(p)^\dagger \, .$$

17. Show:

$$\exp(iPa)\phi(x)\exp(-iPa) = \phi(x+a) \, .$$

Hint: It is sufficient to check the infinitesimal version of this relation.

18. Show:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(x-y)\phi(0)|0\rangle \, .$$

Hint: Use the formula of the previous problem.

19. The propagator of the (free) Klein-Gordon field is defined by

$$\Delta(x) = i\langle 0|T\phi(x)\phi(0)|0\rangle \, .$$

Show: $\Delta(-x) = \Delta(x)$.

20. Show that $\Delta(x)$ (as defined in the previous problem) is a Green function of the Klein-Gordon operator, i.e.

$$(\Box + m^2)\Delta(x) = \delta^{(4)}(x) \, .$$

Discuss the behaviour of $\Delta(x)$ for positive (negative) x^0.

21. The one-dimensional harmonic oscillator is described by the Hamilton operator

$$H = \frac{P(t)^2}{2m} + \frac{m\omega^2 Q(t)^2}{2} .$$

The position operator $Q(t)$ and the momentum operator $P(t)$ fulfil the canonical commutation relation

$$[Q(t), P(t)] = i\hbar \mathbb{1} .$$

   (a) Verify that Heisenberg's equation of motion for $Q(t)$,

   $$\dot{Q}(t) = \frac{i}{\hbar} [H, Q(t)] ,$$

   implies the classical equation of motion $\ddot{Q}(t) + \omega^2 Q(t) = 0$.

   (b) Express $Q(t)$ in terms of the ladder operators a and $a^\dagger$, which satisfy the commutation relation $\left[a, a^\dagger\right] = \mathbb{1}$.

   (c) Calculate the two-point function $\langle 0|TQ(t_1)Q(t_2)|0\rangle$.

22. Determine the generating functional of the one-dimensional harmonic oscillator,

$$Z[f] = \langle 0|Te^{\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, f(t)Q(t)}|0\rangle ,$$

using the path integral representation

$$Z[f] = \frac{1}{\mathcal{N}} \int [dq] \exp\left\{ \frac{i}{\hbar} \int_{-\infty}^{+\infty} dt \left[\frac{m}{2}\dot{q}(t)^2 - \frac{m(\omega^2 - i\varepsilon)}{2} q(t)^2 + f(t)q(t) \right] \right\} .$$

Give a physical interpretation of the external field $f(t)$. Verify the result for the two-point function obtained with the operator method.

Hint: The path integral calculation is completely analogous to the one for a free field, discussed in detail in the lecture. The position variable $q(t)$ can be interpreted as a scalar field living in $0+1$-dimensional spacetime.

23. The generating functional of a free *non-Hermitian* scalar field $\phi(x)$,

$$Z[f] = \langle 0|Te^{i\int d^4x \left(f(x)^*\phi(x) + f(x)\phi(x)^\dagger\right)}|0\rangle ,$$

can be deduced from the generating functionals of two Hermitian scalar fields $\phi_{1,2}(x)$ with equal masses. Use this relation to derive the explicit form of $Z[f]$.

Hint: $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $f = (f_1 + if_2)/\sqrt{2}$, $f_i^* = f_i$.

24. Using the result of the previous problem, discuss the pairing rule for the Green function

$$\langle 0|T\phi(x_1)\ldots\phi(x_m)\phi(y_1)^\dagger\ldots\phi(y_n)^\dagger|0\rangle$$

of a non-Hermitian scalar field $\phi(x)$.

25. Use Noether's theorem to derive the conserved current j^μ associated with the global $U(1)$ symmetry $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ of the Lagrange density $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$.

26. Compute the Gaussian mean values

$$\langle\langle\varphi(x_1)\varphi(x_2)e^{iS_{\rm int}}\rangle\rangle\ , \qquad \langle\langle e^{iS_{\rm int}}\rangle\rangle$$

in φ^4 theory,

$$S_{\rm int} = -\frac{\lambda}{4!}\int d^dy\,\varphi(y)^4\ ,$$

including the contributions of order λ. Convince yourself that the contributions of graphs with vacuum bubbles cancel when the ratio of the two terms is taken.

27. Show the following formula in dimensional regularization ($\alpha, \beta \in \mathbb{N}$):

$$\int\frac{d^dk}{(2\pi)^d}\frac{(k^2)^\beta}{(M^2-k^2-i\varepsilon)^\alpha} = \frac{(-1)^\beta i}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\beta-d/2)\Gamma(\beta+d/2)}{\Gamma(\alpha)\Gamma(d/2)}M^{d+2\beta-2\alpha}\ .$$

Discuss the case $\alpha = 0$ and the implication for $\delta^d(0)$ in dimensional regularization.

28. Write the finite one-loop function ($d = 4$)

$$\bar{B}(p^2, m^2) = B(p^2, m^2) - B(0, m^2)$$

in the form

$$\bar{B}(p^2, m^2) = \int_0^1 d\alpha\, f(\alpha, p^2, m^2)\ .$$

29. Using the previous result, determine the imaginary part of $\bar{B}(p^2, m^2)$.

30. Consider the kinematics of the scattering process $\varphi(p_1)\varphi(p_2) \to \varphi(p_3)\varphi(p_4)$ in the center of mass system. Express the Mandelstam variables

$$t = (p_1 - p_3)^2\ , \quad u = (p_1 - p_4)^2$$

in terms of $s = (p_1 + p_2)^2$ and the scattering angle θ in the center of mass system.

31. Use Noether's theorem to derive the form of the conserved current j^μ associated with the global $U(1)$ symmetry

$$\psi(x) \to e^{i\alpha}\psi(x) \ , \quad \bar{\psi}(x) \to e^{-i\alpha}\bar{\psi}(x)$$

of the Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$.

32. Use the Dirac equation to show that $\partial_\mu j^\mu = 0$ is indeed fulfilled for

$$j^\mu = q\bar{\psi}\gamma^\mu\psi \ .$$

33. The Fourier decomposition of the field operator of a free Dirac field is given by

$$\psi(x) = \sum_s \int d\mu(p) \ \left[b(p,s)u(p,s)e^{-ip\cdot x} + d(p,s)^\dagger v(p,s)e^{ip\cdot x}\right] \ .$$

The creation and annihilation operators fulfil the anticommutation relations

$$\begin{aligned} \left\{b(p,s),\, b(p',s')^\dagger\right\} &= \left\{d(p,s),\, d(p',s')^\dagger\right\} = \delta(p,p')\delta_{ss'} \ , \\ \left\{b(p,s),\, b(p',s')\right\} &= \left\{d(p,s),\, d(p',s')\right\} = 0 \ , \\ \left\{b(p,s),\, d(p',s')\right\} &= \left\{b(p,s),\, d(p',s')^\dagger\right\} = 0 \ . \end{aligned}$$

Using these relations, verify the canonical anticommutaion relations for the field operator:

$$\left\{\psi_a(t,\vec{x}),\, \psi_b(t,\vec{y})^\dagger\right\} = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \ , \quad \left\{\psi_a(x),\, \psi_b(y)\right\}\Big|_{x^0=y^0} = 0 \ .$$

34. Express the energy-momentum operator

$$P^\mu = \int d^3x \ :\psi^\dagger i\partial^\mu\psi:$$

and the charge operator

$$Q = q\int d^3x \ :\psi^\dagger\psi:$$

in terms of creation and annihilation operators.

35. Verify the commutation relations of P^μ and Q with the creation and annihilation operators listed in the lecture notes.

36. Show:

$$i\left[P^\mu, \psi(x)\right] = \partial^\mu\psi(x) \ .$$

Discuss the space-time shift

$$\exp(ia\cdot P)\,\psi(x)\,\exp(-ia\cdot P)$$

following from this relation.

37. The generating functional of the free Dirac field was found to be

$$Z[\eta,\bar\eta] \equiv \left\langle 0 \left| T\exp\left\{ i\int d^4x\ \left[\bar\eta(x)\Psi(x)+\overline{\Psi}(x)\eta(x)\right]\right\}\right| 0\right\rangle$$
$$= \exp\left\{ i\int d^4x\, d^4y\, \bar\eta(x)S(x-y)\eta(y)\right\} \ .$$

Using this formula, compute

$$\langle 0|T\{\Psi_{a_1}(x_1)\overline{\Psi}_{b_1}(y_1)\ldots\Psi_{a_n}(x_n)\overline{\Psi}_{b_n}(y_n)\}|0\rangle \ .$$

38. Compute the two-body phase space integral

$$\int \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0}\delta^{(4)}(P-k_1-k_2)f(k_1\cdot k_2),$$

where $k_i^2 = m_i^2$. P denotes a timelike 4-vector. Give your final answer in terms of the function $\lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$.

Hint: The integral is a Lorentz scalar.

39. Compute

$$I^\mu = \int \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0}\,\delta^{(4)}(P-k_1-k_2)\,k_2^\mu.$$

Hint: The integral is a Lorentz vector and can be written in the form $I^\mu = JP^\mu$, where J is a scalar function (why?).

40. Compute the tensor integral

$$I^{\mu\nu} = \int \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0}\,\delta^{(4)}(P-k_1-k_2)\,k_1^\mu k_2^\nu.$$

41. Compute the invariant amplitude of electron proton scattering ($e^-p\to e^-p$) at tree level. Describe the electromagnetic interaction of the proton by simply using the appropriate covariant derivative in the Dirac Lagrangean of the proton.

Remark: In contrast to the electron, the proton is not a pointlike partice. This reflects itself by the fact that e.g. the magnetic moment of the proton is not correctly desribed by the Dirac Lagrangean alone. In a phenomenological description, an additional term $\sim \bar\psi\sigma_{\mu\nu}\psi F^{\mu\nu}$ is needed. In this exercise, we ignore these complications.

42. Compute the differential cross section of electron proton scattering in the rest frame of the proton, using the result of the previous problem. Determine also the total cross section of this reaction.

43. Verify the Gordon decomposition:

$$\bar{u}(p', s')\gamma^\mu u(p, s) = \bar{u}(p', s')\left[p^\mu + p'^\mu + i\sigma^{\mu\nu}(p' - p)_\nu\right] u(p, s)/(2m).$$

44. Compute $\gamma^\alpha\gamma^\mu\gamma_\alpha$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha$ and $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha$ ($d = 4$).

45. The Higgs ($M_h = 125\,\mathrm{GeV}$) interacts with the elementary fermions by the Yukawa couplings

$$\mathcal{L} = -\sum_f g_f \bar{f}(x) f(x) h(x),$$

where $g_f = m_f/v$ ($v = 246\,\mathrm{GeV}$). Compute the contribution of the virtual Higgs boson to the anomalous magnetic moment of the electron (at one loop).

46. The structure constants f_{abc} of a Lie algebra $\mathcal{L}$ with generators T_a are defined by

$$[T_a, T_b] = i f_{abc} T_c.$$

Show that f_{abc} is totally antisymmetric, if the T_a form an orthogonal basis of $\mathcal{L}$ ($\mathrm{Tr}(T_a T_b) = c\,\delta_{ab}$).

47. Show that $(t_a)_{bc} = -i f_{abc}$ defines a representation of $\mathcal{L}$ (adjoint representation).

48. The transformation formula for a nonabelian gauge field $A_\mu = A_\mu^a T_a$ under a local gauge transformation is given by

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}.$$

Show that the generalized field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

transforms as

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1}.$$

49. The positive frequency part of the massless scalar propagator is given by

$$\Delta_0^+(x) := i\int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{-ip\cdot x}, \quad x^0 \to x^0 - i\varepsilon, \quad p^0 = |\vec{p}\,|.$$

Show that this function can be written in the form

$$\Delta_0^+(x) = \frac{1}{4\pi^2 i[(x^0 - i\varepsilon)^2 - \vec{x}^{\,2}]}.$$

50. Determine the propagator of a real spin 1 field described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A_\mu A^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.