Chapter 10 : Identical Particles

→ Particles with spin are divided into two fundamental classes.

(1) Particles with integer spin ($s = 0, 1, 2, \ldots$) :

It is possible to put an arbitrary number of particles into the same single particle state. (e.g. photons in a laser, Bose-Einstein-Kondensat)
These (integer spin) particles are called **BOSONS**.

(2) Particles with half-integer spin ($s = \frac{1}{2}, \frac{3}{2}, \ldots$) :

It is not possible to put more than 1 particle into the same single particle state.
These (half-integer) particles are called **FERMIONS**.

This different statistical behavior of bosons and fermions can be mathematically proven in the context of relativistic quantum field theory.

10.1. Multiparticle Systems

→ Consider a system made of 2 distinguishable particles (e.g. $e^- - p^+$ system).

General form of the wave function:

$$\psi(\underbrace{\vec{x}_1, \sigma_1}_{\substack{\text{location, spin}\\ \text{of particle 1}}} ; \underbrace{\vec{x}_2, \sigma_2}_{\substack{\text{location, spin}\\ \text{of particle 2}}}) = \langle \vec{x}_1, \sigma_1 ; \vec{x}_2, \sigma_2 | \psi \rangle$$

The state $|\psi\rangle$ is an element of the direct product Hilbert space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, where $\mathcal{H}^{(i)}$ is the single-particle Hilbert space of particle i.

→ Consider a system with 2 indistinguishable particles (e.g. 2 photons, 2 electrons).

(1) Bosons : The wave function is symmetric under the simultaneous exchange of all quantum numbers of particles 1 and 2 :

$$\boxed{\psi(\vec{x}_1, \sigma_1 ; \vec{x}_2, \sigma_2) = \psi(\vec{x}_2, \sigma_2 ; \vec{x}_1, \sigma_1)}$$

The two-particle wave function is totally symmetric.

The Hilbert space $\mathcal{H}_2^B$ is the symmetric tensor product of the two single particle Hilbert spaces:

$$\mathcal{H}_2^B = \mathcal{H}_1 \otimes_S \mathcal{H}_2$$

Basis wave functions for $\mathcal{H}_2^B$.

Let $\{\phi_n(\vec{x},\sigma)\}$ be a CONS of the single particle Hilbert spaces $\mathcal{H}_i$; then each wave function $\psi \in \mathcal{H}_2^B$ can be written as

$$\psi(\vec{x}_1,\sigma_1;\vec{x}_2,\sigma_2) = \sum_{m,n} c_{mn}\, \phi_n(\vec{x}_1,\sigma_1)\, \phi_m(\vec{x}_2,\sigma_2)$$

with $c_{mn} = c_{nm}$.

(2) Fermions: The wave function is antisymmetric under the simultaneous exchange of all quantum numbers of particles 1 and 2:

$$\psi(\vec{x}_1,\sigma_1;\vec{x}_2,\sigma_2) = -\psi(\vec{x}_2,\sigma_2;\vec{x}_1,\sigma_1)$$

The Hilbert space $\mathcal{H}_2^F$ is the anti-symmetric tensor product of the two single particle Hilbert spaces:

$$\mathcal{H}_2^F = \mathcal{H}_1 \otimes_A \mathcal{H}_2$$

The antisymmetry of the wave function for identical fermions under particle quantum number exchange is called **PAULI-PRINCIPLE**.

The two-particle wave function is totally antisymmetric.

We have: $\psi(\vec{x},\sigma;\vec{x},\sigma) = -\psi(\vec{x},\sigma;\vec{x},\sigma) = 0$,

So the wave function of 2 identical fermions (i.e. they have exactly the same quantum numbers such as location, spin direction, spin, ...) is identical zero, which means that that state does not exist.
This principle is called **PAULI EXCLUSION PRINCIPLE**.

Basis wave functions for $\mathcal{H}_2^F$.

Let $\{\phi_n(\vec{x},\sigma)\}$ be a CONS of the single particle Hilbert spaces $\mathcal{H}_i$; then each wave function $\psi \in \mathcal{H}_2^F$ can be written as

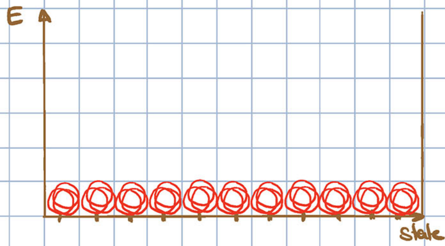
$$\psi(\vec{x}_1,\sigma_1;\vec{x}_2,\sigma_2) = \sum_{m,n} c_{mn}\, \phi_n(\vec{x}_1,\sigma_1)\, \phi_m(\vec{x}_2,\sigma_2)$$

with $c_{mn} = -c_{nm}$ (and in particular $c_{nn} = 0$)

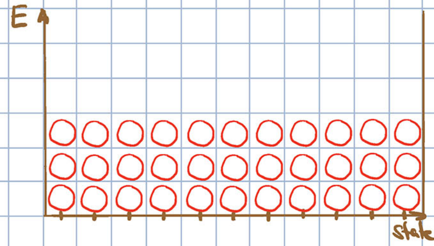
→ In the case of general multiparticle states with N bosons (or fermions) the (anti)symmetrization has to be performed w.r. to the N bosons (fermions).

→ In a bosonic multiparticle system of identical particles all particles can in principle occupy the lowest energy eigenstate (e.g. at temperatures very close to zero).

In a fermionic multiparticle system of identical particles each state can be occupied only at most once, so that the lowest possible energy state of the system is where the particles successively fill up the lowest available energy eigenstates.



bosonic multiparticle system



fermionic multiparticle system

$$\boxed{E^B_{min} < E^F_{min}}$$