Chapter 6: Angular Momentum

6.1. Unitary Transformations

→ We repeat general properties of the unitary operation of spatial translation

$T_{\vec{a}} = \exp(-i\vec{a}\vec{P}/\hbar)$ which can easily cause confusion.

The momentum operator $\vec{P}$ is the generator of translations.

→ Translation acting on a wave function in $\vec{x}$-space:

$$\psi(\vec{x}) \to T_{\vec{a}}\,\psi(\vec{x}) = \exp(-i\vec{a}\vec{P}/\hbar)\,\psi(\vec{x}) = \exp(-\vec{a}\vec{\nabla})\,\psi(\vec{x}) = \psi(\vec{x}-\vec{a})$$

Translation acting on location eigenstates:

$$\Rightarrow \langle\vec{x}|T_{\vec{a}}\,\psi\rangle = \langle\vec{x}|\exp(-i\vec{a}\vec{P}/\hbar)\,\psi\rangle = \langle\vec{x}-\vec{a}|\psi\rangle$$

$$\Rightarrow |\vec{x}\rangle \to T_{\vec{a}}|\vec{x}\rangle = \exp(-i\vec{a}\vec{P}/\hbar)|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$$

Translation acting on momentum eigenstates:

$$|\vec{q}\rangle \to T_{\vec{a}}|\vec{q}\rangle = \exp(-i\vec{a}\vec{P}/\hbar)|\vec{q}\rangle = \exp(-i\vec{a}\vec{q}/\hbar)|\vec{q}\rangle$$

Translation acting on an operator:

Let $|\psi\rangle$ be a state an A a linear operator $\Rightarrow$ $A|\psi\rangle$ is also a state

$$\Rightarrow T_{\vec{a}}A|\psi\rangle = \underbrace{T_{\vec{a}}\,A\,T_{-\vec{a}}}_{\text{translated operator}}\;\underbrace{T_{\vec{a}}|\psi\rangle}_{\text{tranlated state}}$$

$$\hookrightarrow A \to T_{\vec{a}}\,A\,T_{-\vec{a}} = \exp(-i\vec{a}\vec{P}/\hbar)\,A\,\exp(+i\vec{a}\vec{P}/\hbar)$$

Translation acting on a function of the $\vec{X}$ location operator: use $[X_i, P_j] = i\hbar\,\delta_{ij}\,\mathbb{1}$, $[f(X_i), P_j] = i\hbar\,\delta_{ij}\,f'(X_i)$

$$f(\vec{X}) \to \exp(-i\vec{a}\vec{P}/\hbar)\,f(\vec{X})\,\exp(+i\vec{a}\vec{P}/\hbar) = f(\vec{X}-\vec{a}\mathbb{1})$$

→ There is also a translation in momentum space that is generated by the $(-\vec{X})$ operator.

$$T_{\vec{p}}^{\,mom} = \exp(i\vec{p}\vec{X}/\hbar)$$

This can be seen from the fact that $[f(P_i), X_j] = -i\hbar\,f'(P_i)\,\delta_{ij}$ or from the momentum space representation of the $\vec{X}$ operator:

$$(\vec{X})_{p\text{-space}} = +i\hbar\frac{\partial}{\partial\vec{p}}$$

So we have: $\tilde{\Psi}(\vec{p}) \rightarrow \exp(i\vec{q}\vec{X}/\hbar)\,\tilde{\Psi}(\vec{p}) = \tilde{\Psi}(\vec{p}-\vec{q})$

$$|\vec{p}\rangle \rightarrow \exp(i\vec{q}\vec{X}/\hbar)|\vec{p}\rangle = |\vec{p}+\vec{q}\rangle$$

$$|\vec{x}\rangle \rightarrow \exp(i\vec{q}\vec{X}/\hbar)|\vec{x}\rangle = \exp(i\vec{q}\vec{x}/\hbar)|\vec{x}\rangle$$

$$A \rightarrow \exp(i\vec{q}\vec{X}/\hbar)\,A\,\exp(-i\vec{q}\vec{X}/\hbar)$$

$$f(\vec{P}) \rightarrow \exp(i\vec{q}\vec{X}/\hbar)f(\vec{P})\exp(-i\vec{q}\vec{X}/\hbar) = f(\vec{P}-\vec{q}\,\mathbb{1})$$

6.2. Orbital Angular Momentum and Spatial Rotations

→ Orbital angular momentum operator: $\vec{L} = \vec{X}\times\vec{P}$, $L_k = \epsilon_{klm}X_l P_m$ → generator for rotations

There is no issue with ordering of $\vec{X}$ and $\vec{P}$ operators because $\epsilon_{klm}X_l P_m = \epsilon_{klm}P_m X_l$ due to $l \neq m$.

→ Unitary operator for spatial rotation by angle $|\vec{\alpha}|$ around axis $\frac{\vec{\alpha}}{|\vec{\alpha}|}$: $\exp(-i\vec{\alpha}\vec{L}/\hbar)$

Check by an infinitesimal rotation: $\exp(-i\vec{\varepsilon}\vec{L}/\hbar) = \mathbb{1} - i\vec{\varepsilon}\vec{L}/\hbar \quad |\vec{\varepsilon}| \ll 1$

$$\hookrightarrow (\mathbb{1}-i\vec{\varepsilon}\vec{L}/\hbar)|\vec{x}\rangle = (\mathbb{1}-i\varepsilon_k\epsilon_{klm}X_l P_m/\hbar)|\vec{x}\rangle \overset{m\neq l}{=} (\mathbb{1}-i\varepsilon_k\epsilon_{klm}P_m X_l/\hbar)|\vec{x}\rangle$$

$$= (\mathbb{1}-i\varepsilon_k\epsilon_{klm}P_m x_l/\hbar)|\vec{x}\rangle = (\mathbb{1}-i(\vec{\varepsilon}\times\vec{x})\vec{P}/\hbar)|\vec{x}\rangle$$

$$= \exp(-i(\vec{\varepsilon}\times\vec{x})\vec{P}/\hbar)|\vec{x}\rangle$$

$$\overset{\text{Ch. 6.1.}}{=} |\vec{x}+\vec{\varepsilon}\times\vec{x}\rangle \quad \leftarrow \text{indeed correctly rotated state (Ch. 5.2)}$$

$\Rightarrow$ $\exp(-i\vec{\alpha}\vec{L}/\hbar)|\vec{x}\rangle = |R(\vec{\alpha})\vec{x}\rangle$, $R(\vec{\alpha})$: spatial rotation matrix, see Ch. 5.2

The same should also happen to the $|\vec{p}\rangle$ state because a rotation treats all vectors equally:

$$(\mathbb{1}-i\vec{\varepsilon}\vec{L}/\hbar)|\vec{p}\rangle = (\mathbb{1}-i\varepsilon_k\epsilon_{klm}X_l P_m/\hbar)|\vec{p}\rangle = (\mathbb{1}-i\varepsilon_k\epsilon_{klm}X_l p_m/\hbar)|\vec{p}\rangle$$

$$= (\mathbb{1}+i(\vec{\varepsilon}\times\vec{p})\vec{X}/\hbar)|\vec{p}\rangle = \exp(i(\vec{\varepsilon}\times\vec{p})\vec{X}/\hbar)|\vec{p}\rangle$$

$$\overset{\text{Ch. 6.1}}{=} |\vec{p}+\vec{\varepsilon}\times\vec{p}\rangle \quad \leftarrow \text{yes, works!}$$

$\Rightarrow$ $\exp(-i\vec{\alpha}\vec{L}/\hbar)|\vec{p}\rangle = |R(\vec{\alpha})\vec{p}\rangle$, $R(\vec{\alpha})$: spatial rotation matrix

The rotation matrices $R(\vec{\alpha})$ form the group SO(3), which are the set of orthogonal 3×3 real matrices with determinant 1. ↳ $R^T(\vec{\alpha}) = R^{-1}(\vec{\alpha})$

We also check the action of the rotation operator on the $\vec{X}$ and $\vec{P}$ operators:

$$(1 - i\vec{\varepsilon}\,\vec{L}/\hbar)\,\vec{X}\,(1 + i\vec{\varepsilon}\,\vec{L}/\hbar) = (1 - i\vec{\varepsilon}\,\vec{L}/\hbar)\,\vec{X}\,(1 + i\,\varepsilon_k\,\epsilon_{klm} X_l P_m/\hbar) \qquad \text{use } [X_i, P_k] = i\hbar\delta_{ik}\,1$$

$$= \underbrace{(1 - i\vec{\varepsilon}\,\vec{L}/\hbar)(1 + i\vec{\varepsilon}\,\vec{L}/\hbar)}_{1 + O(\varepsilon^2)}\,\vec{X} + \underbrace{(1 - i\vec{\varepsilon}\,\vec{L}/\hbar)(-\vec{\varepsilon}\times\vec{X})}_{O(\varepsilon^2)} \qquad \leftarrow \text{neglect } O(\varepsilon^2) \text{ terms}$$

$$= \vec{X} - \vec{\varepsilon}\times\vec{X} \qquad \leftarrow \text{rotation in negative direction}$$

$$\Rightarrow \exp(-i\vec{\alpha}\vec{L}/\hbar)\,\vec{X}\,\exp(i\vec{\alpha}\vec{L}/\hbar) = R^{-1}(\vec{\alpha})\,\vec{X}$$

obvious to see

$$\exp(-i\vec{\alpha}\vec{L}/\hbar)\,f(\vec{X})\,\exp(i\vec{\alpha}\vec{L}/\hbar) = f(R^{-1}(\vec{\alpha})\,\vec{X})$$

same considerations

$$\exp(-i\vec{\alpha}\vec{L}/\hbar)\,\vec{P}\,\exp(i\vec{\alpha}\vec{L}/\hbar) = R^{-1}(\vec{\alpha})\,\vec{P}$$

$$\exp(-i\vec{\alpha}\vec{L}/\hbar)\,f(\vec{P})\,\exp(i\vec{\alpha}\vec{L}/\hbar) = f(R^{-1}(\vec{\alpha})\,\vec{P})$$

same considerations

$$\exp(-i\vec{\alpha}\vec{L}/\hbar)\,\vec{L}\,\exp(i\vec{\alpha}\vec{L}/\hbar) = R^{-1}(\vec{\alpha})\,\vec{L}$$

$$\exp(-i\vec{\alpha}\vec{L}/\hbar)\,f(\vec{L})\,\exp(i\vec{\alpha}\vec{L}/\hbar) = f(R^{-1}(\vec{\alpha})\,\vec{L})$$

This makes sense since e.g. $f(R^{-1}(\vec{\alpha})\vec{X})$ means that we have to rotate $\vec{X}$ with $R(\vec{\alpha})$ to obtain the same result $f(\vec{x})$ as before the rotation. So the function f is indeed rotated by angle $|\vec{\alpha}|$ around axis $\frac{\vec{\alpha}}{|\vec{\alpha}|}$.
→ Compare to $f(\vec{x}) \to f(\vec{x}-\vec{a})$ which means translation of f by $+\vec{a}$.

The above relations tell that $\vec{L}$ generates the same kind of rotations on any vector operator.
This can also be expressed in the following commutation relations:

$$[L_k, X_l] = i\hbar\,\epsilon_{klm} X_m$$

$$[L_k, P_l] = i\hbar\,\epsilon_{klm} P_m$$

$$[L_k, L_l] = i\hbar\,\epsilon_{klm} L_m$$

Finally we also check the action of the rotation operator on a wave function:

$$\langle \vec{x}|\underbrace{\exp(-i\vec{\alpha}\vec{L}/\hbar)|\psi\rangle}_{\text{state rotated by } |\vec{\alpha}| \text{ around } \frac{\vec{\alpha}}{|\vec{\alpha}|}} = \langle R^{-1}(\vec{\alpha})\vec{x}|\psi\rangle = \psi(R^{-1}(\vec{\alpha})\vec{x}) \qquad \leftarrow \text{indeed function rotated by } R(\vec{\alpha})$$

Infinitesimal form:

$$(\vec{\varepsilon}\times\vec{x})\cdot\vec{\nabla} = \epsilon_{klm}\,\varepsilon_l x_m \partial_k$$
$$= \epsilon_{mkl}\,\varepsilon_l x_m \partial_k = \vec{\varepsilon}\,(\vec{x}\times\vec{\nabla})$$

$$\langle \vec{x}|1 - \tfrac{i}{\hbar}\vec{\varepsilon}\cdot\vec{L}|\psi\rangle = \langle \vec{x} - \vec{\varepsilon}\times\vec{x}|\psi\rangle = \psi(\vec{x} - \vec{\varepsilon}\times\vec{x})$$

$$= \psi(\vec{x}) - (\vec{\varepsilon}\times\vec{x})\,\vec{\nabla}\psi(\vec{x}) = \psi(\vec{x}) - \vec{\varepsilon}(\vec{x}\times\vec{\nabla})\psi(\vec{x}) = \psi(\vec{x}) - \frac{i}{\hbar}\underbrace{\left(\vec{x}\times\frac{\hbar}{i}\vec{\nabla}\right)}_{=\vec{L}}\psi(\vec{x})$$

$$\Rightarrow \langle \vec{x}|\vec{L}|\psi\rangle = \vec{x}\times\frac{\hbar}{i}\vec{\nabla}\,\langle \vec{x}|\psi\rangle = \vec{X}\times\vec{P}\,\langle \vec{x}|\psi\rangle \quad \checkmark \quad \text{o.k.}$$

6.3. General Theory of the Angular Momentum

→ For any quantum mechanical system spatial relations are described by the unitary operators

$\exp(-i\vec{\alpha}\vec{J}/\hbar)$, where $\vec{J}$ is the spatial angular momentum operator, which is Hermitian

$\vec{J} = (J_x, J_y, J_z) = (J_1, J_2, J_3)$

These operators satisfy the angular momentum commutation relation $[J_k, J_l] = i\hbar \underbrace{\epsilon_{klm}}_{\text{"SU(2) structure constants"}} J_m$

It is convenient to consider the operators $T_k \equiv J_k/\hbar$ to avoid factors of $\hbar$ which satisfy

$[T_k, T_l] = i\epsilon_{klm} T_m$ (SU(2) commutation relations)

→ Different realizations of the angular momentum operators $\vec{J}$ (or $\vec{T}$) are called representations.

Examples: (1) Spin-$\frac{1}{2}$ systems: $T_k = \sigma_k/2$

(2) Spin-less particle in $\mathbb{R}^3$: $T_k = -i\epsilon_{klm} x_l \nabla_m$

(3) Spin-$\frac{1}{2}$ particle in $\mathbb{R}^3$: $T_k = (\sigma_k/2) \otimes (-i\epsilon_{klm} x_l \nabla_m)$ $\otimes$ direct product

(1) and (2) are examples for irreducible representations, which are representations that cannot be written as a direct product of more elementary representations.

It is possible to classify the irreducible representations by the eigenvalue of the operator $\vec{T}^2 = T_1^2 + T_2^2 + T_3^2$,
which commutes with each T_k: $[\vec{T}^2, T_k] = 0$

→ General structure of the irreducible representations of the rotation group

We derive the general structure of irreducible representations where we use the eigenvalue of $\vec{T}^2$ to classify the representation and one of the T_k to label each state in the representation. (We take T_3.)
(e.g. Spin-$\frac{1}{2}$: $\vec{T}_k^2 = \frac{1}{4} s(s+1)$ with $s=\frac{1}{2}$, T_z has eigenvalues $\pm\frac{1}{2}$)

We define: $T_\pm = T_1 \pm iT_2 = T_x \pm iT_y$

$$\Rightarrow \quad (T_\pm)^+ = T_\mp \qquad (a)$$
$$[T_z, T_\pm] = iT_y \pm T_x = \pm T_\pm \qquad (b)$$
$$[T_+, T_-] = -2i[T_x, T_y] = 2T_z \qquad (c)$$
$$[\vec{T}^2, T_\pm] = 0 \qquad (d)$$
$$T_+T_- = L_x^2 + L_y^2 - i[T_x, T_y] = T_x^2 + T_y^2 + T_z \qquad (e)$$
$$\hookrightarrow \vec{T}^2 = T_x^2 + T_y^2 + T_z^2 = T_+T_- - T_z + T_z^2 = T_-T_+ + T_z + T_z^2 \qquad (f)$$

Step 1: The eigenvalues of $\vec{T}^2$ are ≥ 0, which we therefore can write as $j(j+1)$, $j \geq 0$
On the eigenspace of $\vec{T}^2$ to the eigenvalue $j(j+1)$ the eigenvalues m of T_z satisfy $m^2 \leq j(j+1)$.
We can use j and m to label each angular momentum state. So we define states $|\psi_{jm}\rangle$ with

$$\vec{T}^2|\psi_{jm}\rangle = j(j+1)|\psi_{jm}\rangle \quad , \quad T_z|\psi_{jm}\rangle = m|\psi_{jm}\rangle \quad , \quad \langle\psi_{jm}|\psi_{jm}\rangle = 1$$

For all states $|\psi\rangle$: $\underbrace{\langle\psi|\vec{T}^2|\psi\rangle}_{j(j+1)} = \overset{\geq 0}{\langle\psi|T_x^2|\psi\rangle} + \overset{\geq 0}{\langle\psi|T_y^2|\psi\rangle} + \underbrace{\overset{\geq 0}{\langle\psi|T_z^2|\psi\rangle}}_{m^2} \geq 0$ (also true to eigenstates) ✓
with $\langle\psi|\psi\rangle = 1$:

Step 2: The eigenspace of $\vec{T}^2$ to the eigenvalue $j(j+1)$ is closed w.r. to the operators T_x, T_y, T_z.
So the eigenspace is also closed w.r. to rotations, which are functions of T_x, T_y, T_z.

Let $|\psi\rangle$ be an eigenstate to $\vec{T}^2$ with eigenvalue $j(j+1)$, then $\vec{T}^2 T_k|\psi\rangle = T_k\vec{T}^2|\psi\rangle = j(j+1)T_k|\psi\rangle$ ✓

Step 3: The operators $T_\pm$ raise/lower the eigenvalue of T_z by one unit
We have

$$T_\pm|\psi_{jm}\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|\psi_{j\,m\pm 1}\rangle$$

It is possible to define phase factors that way

$T_z T_\pm |\psi_{jm}\rangle \overset{(6)}{=} \pm T_\pm|\psi_{jm}\rangle + T_\pm T_z|\psi_{jm}\rangle = (m\pm 1)\,T_\pm|\psi_{jm}\rangle$ ✓

$\langle T_\pm\psi_{jm}|T_\pm\psi_{jm}\rangle = \langle\psi_{jm}|T_\mp T_\pm\psi_{jm}\rangle$
$\overset{(4)}{=} \langle\psi_{jm}|(\vec{T}^2 - T_z^2 \mp T_z)\psi_{jm}\rangle = j(j+1) - m(m\pm 1)$ ✓

Step 4: The maximal size of eigenvalues m on a j-eigenspace is j : $|m| = j$

Let $m_{max/min}$ be the largest/smallest eigenvalue of T_z on the j-eigenspace, the relation
$T_\pm|\psi_{j\,m_{max/min}}\rangle = \sqrt{j(j+1) - m_{max/min}(m_{max/min}\pm 1)}\;|\psi_{j\,m_{max/min}\pm 1}\rangle$
does lead to a contradiction unless $m_{max} = j$ and $m_{min} = -j$.

Step 5: The results from steps 1-4 are only consistent if

$j = 0, 1, 2, \ldots$ (integer spin)
or
$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ (half integer spin)

and $m = \underbrace{-j, -j+1, \ldots, j-1, j}_{2j+1 \text{ values}}$

For each allowed value of j the $2j+1$ orthonormal states $|\psi_{jm}\rangle$ $(m = -j, \ldots, j)$ form the basis of a subspace of $\mathcal{E}$ that is invariant under rotations and represents a system with spin-j.

Each spin-j system forms an irreducible representation of the $SU(2)$ rotation group

For the angular momentum operators $\vec{J}$ we have: $|\psi_{jm}\rangle \equiv |j,m\rangle$

$$\vec{J}^2|j,m\rangle = \hbar j(j+1)|j,m\rangle \quad , \quad J_z = \hbar m|j,m\rangle$$

$$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\;|j,m\pm 1\rangle$$

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6.4. Examples for Irreducible Representations

→ **Spin-0 (Trivial) Representation**

$j=0\,,\quad T_k=0$ → Particles without spin: e.g. Higgsboson, Mesons (Pions, kaons, ...)

→ **Spin-$\frac{1}{2}$ Representation** → "fundamental representation" of SU(2)

$j=\frac{1}{2}\,,\quad T_k=\frac{1}{2}\sigma_k\,,\quad m=\frac{1}{2},-\frac{1}{2}$

$\vec{T}^2=\frac{3}{4}\mathbb{1}=\frac{1}{2}\left(\frac{1}{2}+1\right)\mathbb{1}$

$T_3=\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}\qquad T_+=T_1+iT_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad T_-=T_1-iT_2=\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$|\frac{1}{2},+\frac{1}{2}\rangle\leftrightarrow\begin{pmatrix}1\\0\end{pmatrix}\qquad T_3|\frac{1}{2},\frac{1}{2}\rangle=\frac{1}{2}|\frac{1}{2},\frac{1}{2}\rangle\qquad T_+|\frac{1}{2},\frac{1}{2}\rangle=0\qquad T_-|\frac{1}{2},\frac{1}{2}\rangle=|\frac{1}{2},-\frac{1}{2}\rangle$

$|\frac{1}{2},-\frac{1}{2}\rangle\leftrightarrow\begin{pmatrix}0\\1\end{pmatrix}\qquad T_3|\frac{1}{2},-\frac{1}{2}\rangle=-\frac{1}{2}|\frac{1}{2},-\frac{1}{2}\rangle\qquad T_+|\frac{1}{2},-\frac{1}{2}\rangle=|\frac{1}{2},+\frac{1}{2}\rangle\qquad T_-|\frac{1}{2},-\frac{1}{2}\rangle=0$

→ **Spin-1 Representation** → 3-dimensional representation, "fundamental representation" of SO(3)

$j=1\,,\quad \vec{T}^2=2\mathbb{1}=1(1+1)\mathbb{1}\,,\quad m=+1,0,-1$

Spin basis: $\{|1,m\rangle\}=\{|1,+1\rangle,|1,0\rangle,|1,-1\rangle\}$ with $T_3|1,m\rangle=m|1,m\rangle$

$T_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\qquad T_+=\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}\qquad T_-=\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}$

$T_1=\frac{1}{2}(T_++T_-)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\qquad T_2=\frac{1}{2i}(T_+-T_-)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$

The spin basis is useful for the description of spin-1 particles in analogy to the description of spin-$\frac{1}{2}$. In the spin basis the 3 basis states $|1,m\rangle$ $(m=1,2,3)$ do not describe the 3 spatial directions (x,y,z). However, there is an alternative basis where the 3 basis states indeed correspond to the spatial directions. This basis is called the "**adjoint representation of SU(2)**".

Adjoint Representation:

The adjoint representation is obtained from the SU(2) structure constants and can be derived from the infinitesimal rotations of spatial 3-vectors.

↳ Rotation by $|\vec{\varepsilon}|\ll 1$ around axis $\frac{\vec{\varepsilon}}{|\vec{\varepsilon}|}$: $\quad \vec{x}\to\vec{x}+\vec{\varepsilon}\times\vec{x}=(\mathbb{1}-i\,\vec{\varepsilon}\,\underbrace{\vec{t}}_{\text{generators of the adjoint rep}})\,\vec{x}$

$$x_n\to x_n+\epsilon_{nek}\,\varepsilon_e\,x_k=\Big(\delta_{nk}-i\,\varepsilon_e\overbrace{[-i\epsilon_{enk}]}^{=(t_{nk})_e}\Big)x_k$$

$$\Rightarrow\quad t_1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\qquad t_2=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}\qquad t_3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$

One can show that: $T_k = U\, t_k\, U^+$ with $U = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & 1 \\ 1 & i & 0 \end{pmatrix}$

This means that a spatial 3-vector $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ has the form $U\vec{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} -x+iy \\ z \\ x+iy \end{pmatrix}$ in the spin-1 basis.

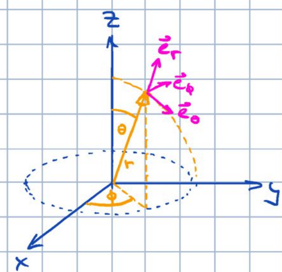
→ Comment:

All representations of angular momentum with half-integer j (i.e. $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$) are not representations of the spatial rotation group $SO(3)$. This is only possible for integer j (i.e. $j = 0, 1, 2, \ldots$).

6.5. Spherical Harmonic Functions

→ The spherical harmonic functions are the angular momentum analogue to the eigenfunctions $\langle \vec{x} | \vec{p} \rangle = \frac{e^{i\vec{p}\vec{x}}}{(2\pi)^{3/2}}$ of the momentum operator $\vec{P}$.
The spherical harmonic functions are an explicit realization of the irreducible representations of angular momentum and the rotation group for all integer values of j, i.e. for: $j = 0, 1, 2, \ldots$

→ We have to switch from cartesian to spherical (polar) coordinates:



$$x = x_1 = r \sin\theta \cos\phi \;,\; y = x_2 = r \sin\theta \sin\phi \;,\; z = x_3 = r\cos\theta$$

In sperical coordinates the basis vectors at each point $\vec{x} = r\vec{e}_r$ in space depend on the (r, θ, ϕ) = (radius, polar angle, azimuthal angle) values of that point $\vec{x}$.

$$\vec{e}_r = \vec{e}_x \sin\theta\cos\phi + \vec{e}_y \sin\theta \sin\phi + \vec{e}_z \cos\theta$$

$\theta \to \theta + \frac{\pi}{2}$

$$\vec{e}_\theta = \vec{e}_x \cos\theta\cos\phi + \vec{e}_y \cos\theta\sin\phi - \vec{e}_z \sin\theta$$

$\theta = \frac{\pi}{2}$, $\phi \to \phi + \frac{\pi}{2}$

$$\vec{e}_\phi = -\vec{e}_x \sin\phi + \vec{e}_y \cos\phi$$

Form of the nabla: $\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

$\vec{e}_r \times \vec{e}_\theta = \vec{e}_\phi$
$\vec{e}_r \times \vec{e}_\phi = -\vec{e}_\theta$

Angular momentum operator:

$$\vec{L} = \vec{x} \times \frac{\hbar}{i}\vec{\nabla} = \frac{\hbar}{i}\, r\, \vec{e}_r \times \vec{\nabla} = \frac{\hbar}{i}\left(\vec{e}_\phi \frac{\partial}{\partial\theta} - \vec{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \right)$$

↳
$$\left.\begin{aligned} L_x = L_1 &= \frac{\hbar}{i}\left(-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right) \\ L_y = L_2 &= \frac{\hbar}{i}\left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ L_z = L_3 &= \frac{\hbar}{i}\frac{\partial}{\partial\phi} \end{aligned}\right\}$$

$$L_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$d\Omega = d\cos\theta\, d\phi = \sin\theta\, d\theta\, d\phi$$

→ The angular momentum operators L_k only act on the polar and azimuthal angles.
So they act on functions $Y(\theta,\phi)$ defined on the surface of the unit sphere (called S^2)

The set of complex-valued functions defined on the surface of the unit sphere

$$L^2(S^2) = \left\{ Y(\theta,\phi): S^2 \to \mathbb{C} \;\middle|\; \int d\Omega \, |Y(\theta,\phi)|^2 = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, |Y(\theta,\phi)|^2 < \infty \right\}$$

with the scalar product $\langle Y_1 | Y_2 \rangle = \int_{S^2} d\Omega \, Y_1^*(\Omega) Y_2(\Omega)$, $\Omega = (\theta,\phi)$ is a Hilbert space.

→ The spherical harmonic functions $Y_{\ell m}(\theta,\phi) \in L^2(S^2)$ are the simultaneous eigenfunctions of the operators $\vec{L}^2$ and L_z with

$$\vec{L}^2 Y_{\ell m}(\Omega) = \hbar^2 \ell(\ell+1) Y_{\ell m}(\Omega) , \quad \ell = 0,1,2,\dots$$

$$L_z Y_{\ell m}(\theta,\phi) = \hbar m Y_{\ell m}(\theta,\phi) , \quad m = -\ell, -\ell+1, \dots, \ell-1, \ell$$

and they form a CONS of $L^2(S^2)$.

We determine the $Y_{\ell m}(\theta,\phi)$ by using the separation ansatz $Y_{\ell m}(\theta,\phi) = P_{\ell m}(\cos\theta)\, b_m(\phi)$ and starting with the eigenvalue equation

$$L_z Y_{\ell m} = \hbar m Y_{\ell m} \iff \frac{\partial}{\partial\phi} b_m(\phi) = i m\, b_m(\phi) \Rightarrow b_m(\phi) = e^{im\phi}$$

Thus the second eigenvalue equation $\vec{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$ can be rewritten as

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) - \frac{m^2}{\sin^2\theta} + \ell(\ell+1) \right] P_{\ell m}(\cos\theta) = 0$$

We change variables : $\xi = \cos\theta \Rightarrow \frac{\partial}{\partial\theta} = \frac{d\cos\theta}{d\theta}\frac{\partial}{\partial\xi} = -\sin\theta \frac{\partial}{\partial\xi}$ $\quad (-1 \le \xi \le 1)$

$$\Rightarrow \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) = \frac{\partial}{\partial\xi}\left(\sin^2\theta \frac{\partial}{\partial\xi}\right) = \frac{\partial}{\partial\xi}\left((1-\xi^2)\frac{\partial}{\partial\xi}\right) = (1-\xi^2)\frac{\partial^2}{\partial\xi^2} - 2\xi\frac{\partial}{\partial\xi}$$

$$\Rightarrow \left[(1-\xi^2)\frac{d^2}{d\xi^2} - 2\xi\frac{d}{d\xi} + \ell(\ell+1) - \frac{m^2}{1-\xi^2} \right] P_{\ell m}(\xi) = 0$$

This is the defining equation of the associated Legendre polynomials.
The $P_{\ell m}$ can be obtained from the (regular) Legendre polynomials $P_\ell(\xi) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\xi^\ell}(\xi^2-1)^\ell$, $\ell = 0,1,\dots$
by the relation

$$P_{\ell m}(\xi) = (1-\xi^2)^{\frac{m}{2}} \frac{d^m}{d\xi^m} P_\ell(\xi) \quad \text{with } m > 0.$$

The regular Legendre polynomials form a complete set of orthogonal functions on the interval $[-1,1]$ with $\int_{-1}^{1} d\xi \, P_n(\xi) P_m(\xi) = \frac{2}{2n+1}\delta_{nm}$
The satisfy the recursion relations

$$(\ell+1)P_{\ell+1}(\xi) = (2\ell+1)\xi P_\ell(\xi) - \ell P_{\ell-1}(\xi) , \quad (1-\xi^2)\frac{d}{d\xi}P_\ell(\xi) = \ell\left(P_{\ell-1}(\xi) - \xi P_\ell(\xi) \right)$$

Lowes Legendre polynomials: $P_0(\xi) = 1$, $P_1(\xi) = \xi$, $P_2(\xi) = \frac{1}{2}(3\xi^2-1)$, $P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi)$, ...

The associated Legendre polynomials have the properties

$$\int_{-1}^{1} d\xi\, P_{\ell m}(\xi)\, P_{\ell' m}(\xi) = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'} \quad (m \geq 0)$$

$$P_{\ell m}(-\xi) = (-1)^{\ell+m} P_{\ell m}(\xi)$$

$$P_{\ell 0}(\xi) = P_\ell(\xi) \quad , \quad P_{\ell\ell}(\xi) = (2\ell-1)!!\,(1-\xi^2)^{\frac{\ell}{2}} \qquad (2\ell-1)!! := (2\ell-1)(2\ell-3)\dots 3\cdot 1$$

→ Explicit form of the spherical harmonic functions:

$$Y_{\ell m}(\theta,\phi) = (-1)^{\frac{(m+|m|)}{2}}\left[\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{1/2} P_{\ell|m|}(\cos\theta)\, e^{im\phi}$$

Orthonormality: $\int d\Omega\, Y^*_{\ell m}(\Omega) Y_{\ell' m'}(\Omega) = \delta_{\ell\ell'}\delta_{mm'}$

Completeness: $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta',\phi') = \delta(\cos\theta-\cos\theta')\delta(\phi-\phi') = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi')$

Addition theorem: $\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi) Y_{\ell m}(\theta',\phi') = \frac{2\ell+1}{4\pi} P_\ell(\cos\alpha)$, α: angle between directions (θ,ϕ) and (θ',ϕ')

$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

Parity transformation: $\vec{x} \to P\vec{x} = -\vec{x} \Rightarrow (\theta,\phi) \to (\pi-\theta, \phi+\pi)$

$$P\, Y_{\ell m}(\theta,\phi) = Y_{\ell m}(\pi-\theta,\phi+\pi) = (-1)^\ell Y_{\ell m}(\theta,\phi) \qquad \cos(\pi-\theta) = -\cos\theta$$

m-Symmetry: $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$

Some explicit expressions: (use m-symmetry to obtain expressions for $m<0$)

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \quad , \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}, \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}, \quad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

→ The completeness property of the spherical harmonic functions $Y_{\ell m}$ allows to write every wave function $\psi(\vec{x}) \in L^2(\mathbb{R}^3)$ as

$$\psi(\vec{x}) = \psi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} u_{\ell m}(r) Y_{\ell m}(\theta,\phi) \quad \text{where}$$

$$u_{\ell m}(r) = \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\phi\, Y^*_{\ell m}(\theta,\phi)\psi(\vec{x}) = \int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\phi\, Y^*_{\ell m}(\theta,\phi)\psi(\vec{x})$$

angular / multipole spectrum

06/11/2016 ⑤

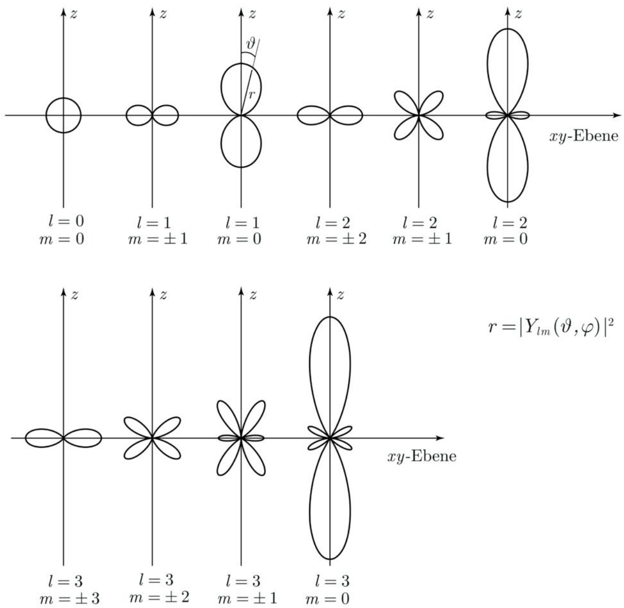


Abb. 5.6. Polardiagramme der Bahndrehimpulseigenfunktionen Y_{lm} mit $l = 0, 1, 2, 3$

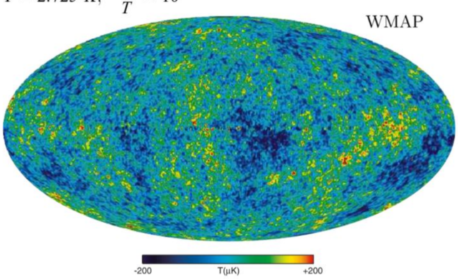
Naming scheme:

$l=0$ state: s-orbital
$l=1$ states: p-orbitals
$l=2$ states: d-orbitals
$l=3$ states: f-orbitals

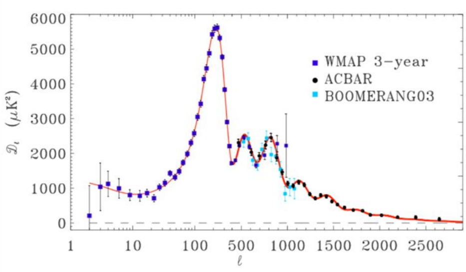
CMB = Cosmic Microwave Background

What do we learn from CMB observations - Rubakov, V.A. *et al.* Phys.Atom.Nucl. 75 (2012) 1123-1141 arXiv:1008.1704 [astro-ph.CO] ITEP-TH-30-10

$T = 2.725°K, \quad \frac{\delta T}{T} \sim 10^{-5}$



~The CMB temperature map, obtained by WMAP experiment \cite{WMAP}. The brighter a region is, the hotter radiation comes from it.



~The angular spectrum of the CMB temperature anisotropy \cite{ACBAR}. The line is a prediction of the standard ΛCDM model. The quantity in vertical axis is D_l defined by (???).