

# Quantum Mechanics I - Lecture Notes

## Chapter 1: Introduction

### 1.1 Historical Overview

→ The beginning of quantum physics.

→ First essential steps at the end of the 19th and the beginning of the 20th century

#### (1) Particle Character of Light

# = number

##### Max Planck (1900)

→ Introduction of the "Planck constant"  $\hbar$  to find correct theoretical description of the energy density of black-body radiation.

↳ energy density ( $E_{\nu}/\text{sr}$ ) of e.m. radiation in frequency interval  $[\nu, \nu + d\nu]$ :

$$u(\nu) d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

Planck's Radiation Law

(valid in thermal equilibrium)

Derivation: (a) Density of e.m. radiation starts: → consider box

↳ # of modes in freq. interval  $[\nu, \nu + d\nu]$  in box  
 V of box

$$= g(\nu) d\nu = \frac{4\pi}{c^3} V^2 d\nu$$



only certain wavelengths fit into box with volume V

↳ # of states with given  $\nu$  is countable

Boltzmann constant

(b) classic: each state can carry an arbitrary amount of energy  $\Rightarrow E_{\nu}(v, T) = h\nu$

quantum: each state can carry energy in packets of  $\hbar\nu$

↳ Boltzmann: probability that a state with energy  $E$  is created at temperature  $T$

$$p(E) \propto e^{-E/kT}$$

↳ mode with freq.  $\nu$  can carry in packets with  $n = 0, 1, 2, \dots$

$$\Rightarrow p(n) = e^{-n\nu/kT} (1 - e^{-\nu/kT})^n$$

$$\Rightarrow \text{average energy } E(v, T) = \sum_{n=0}^{\infty} n h\nu g(n) = \frac{h\nu}{e^{h\nu/kT} - 1} \xrightarrow[n \rightarrow \infty]{kT} \text{ (classic)}$$

$$\Rightarrow u(\nu) = 2 E(v, T) g(\nu) \xrightarrow[2 \text{ polarizations}]{\nu \rightarrow 0} u_{\text{el}}(\nu) = \frac{8\pi h\nu^2}{c^3} \xrightarrow{\nu \rightarrow \infty} \text{ Rayleigh-Jeans Law}$$

$$\rightarrow \int_0^{\infty} d\nu u(\nu) d\nu = \frac{8\pi^4 \nu^5}{15 c^3 h^2} \text{ total energy density finite and } \propto T^4 \quad (\text{Stefan-Boltzmann-Law})$$

$$\text{but } \int_0^{\infty} d\nu u_{\text{el}}(\nu) = \infty \quad (\text{divergent for large frequencies}) \quad \text{"ultraviolet catastrophe"}$$

→ Planck does not consider the energy quantization to be a property of light, but a property of the matter that radiates or absorbs light. Quantization a mathematical trick that would likely be replaced by a better classic understanding of matter.

→  $u(\nu)$  fits perfectly the experimental observations:  $\hbar = 4.136 \cdot 10^{-34} \text{ eV s}$  Planck constant  
 $\hbar = \frac{h}{2\pi} = 6.582 \cdot 10^{-34} \text{ eV s}$  reduced Planck const.

Albert Einstein (1905)

$$\omega = 2\pi\nu = \frac{2\pi c}{\lambda}$$

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→ Light quantization hypothesis: light consists of particles with energy  $E = h\nu = \hbar\omega$   $\omega = 2\pi\nu$  angular frequency

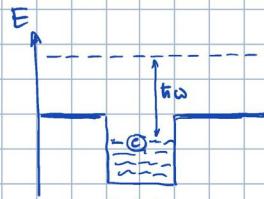
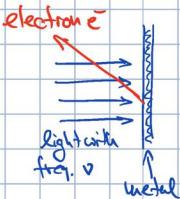
↳  $c$ : speed of light  $\Rightarrow \lambda = \frac{c}{\nu}$  wave length  $|\vec{k}| = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$  (angular) wave number

↳ light particle of freq.  $\nu$  has momentum  $\vec{p} = \hbar\vec{k}$  "photon  $j$ "

and energy  $E = c|\vec{p}| = c\hbar|\vec{k}| = \frac{2\pi c \hbar \nu}{\lambda} = h\nu = \hbar\omega$   
E-p relation of massless particles

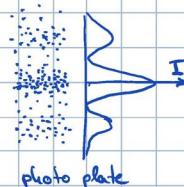
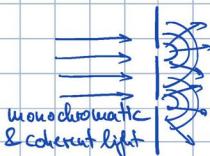
→ Light state of freq.  $\nu$  with energy  $h\nu$   $\hat{=}$   $n$  light particles

→ Photo electric effect:



→ Minimal frequency  $\nu$  required that electrons can overcome a potential barrier, independent of intensity of the light.

→ Double slit experiment



→ At low intensity one observes single hits on the photo plate

→ Compton Effect



→ Light momentum and energy as function of scattering angle  $\theta$  fully explained by particle interpretation

## (2) Atomic Structure

Spectroscopy (since 1905): Atoms absorb and emit light with (atom) specific frequencies (spectral lines)

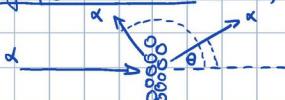
Joseph John Thomson (1903): Plum pudding model



→ Classic stable model of atoms consisting of electrons immersed in some homogeneous positive charge distribution.

→ Cannot explain atom structures

Hans Geiger, Ernest Marsden (1908)



→ Scattering of  $\alpha$ -particles off (thin foil of) gold atoms  
→ Scattering with  $\theta > \frac{\pi}{2}$  (scattering angle) observed which is not compatible with plum pudding model

Ernest Rutherford (1911) → Classical calculation of differential cross section of 2 point charges

$$\vec{r} \rightarrow \text{Coulomb potential } V(\vec{r}) = \frac{2e}{|\vec{r}|} \quad \frac{d\sigma}{d\Omega} = \left( \frac{m_1 m_2 e^2}{2 \pi^2 \sin^2(\frac{\theta}{2})} \right)^2$$

↑ negative charge not relevant for central hits

→ Explains experiment of Geiger/Marsden

↪ Atom: almost all mass concentrated in very small nucleus ( $\sim 10^{-16}$  m) which carries positive charge  $Ze$ ; nucleus surrounded by  $Ze$  electrons

→ Atom  $\leftrightarrow$  planetary system

→ But: stable atoms cannot be described by classic mechanics + E-dynamics (electrons radiate photons, loose energy)  
→ no explanation for spectral lines

Niels Bohr (1913) → Spectral lines explained by transitions (with photon radiation) between discrete energy levels

$$2 \Rightarrow h\nu_{\text{radiation}} = E_n - E_m$$

→ Bohr's H-atom model: classic mechanics + E-dynamics supplemented by "quantum" conditions for allowed  $c$ -orbits

$$\Rightarrow E_n^{\text{bind}} = -\frac{m_e c^2 \alpha^2}{2 n^2} = -\frac{13.6 \text{ eV}}{n^2}, n = 1, 2, 3, \dots \quad \alpha = \frac{e^2}{4 \pi \epsilon_0 c} \approx \frac{1}{137} \quad \text{Sommerfeld's fine structure constant}$$

### (3) Wave-Character of Matter Particles

Louis de Broglie (1923): Every particle has wave properties (just as photons)

→ connection momentum  $\leftrightarrow$  wave number vector:  $\vec{p} = \hbar \vec{k}$  ( $|\vec{k}| = \frac{2\pi}{\lambda}$ ) relativistic E-p relation

→ connection Energy  $\leftrightarrow$  angular frequency:  $E = c \sqrt{\vec{p}^2 + m^2 c^2} = c \sqrt{\hbar^2 k^2 + m^2 c^2} = \hbar \omega$

→ Description of a particle with momentum  $\vec{p} = \hbar \vec{k}$  and mass  $m$  by monochromatic plane wave

$$\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

### (4) Quantum Theory

Werner Heisenberg (1925): Radical departure from concepts of classical mechanics!

→ Position and momentum of a particle related to  $n$ -dimensional matrices (operators)  $Q$  and  $P$ , which stand for the outcome of a measurement of position and momentum.  $P$  and  $Q$  satisfy the

$$\text{Commutation relation } QP - PQ = i\hbar \mathbb{I}$$

→ Development of a matrix mechanics (Heisenberg, Max Born, Pascual Jordan)

Wolfgang Pauli (1925): Solution of H-atom using matrix mechanics: binding energy levels  $E_n$

Erwin Schrödinger (1926):

→ Formulation of a Wave (differential) equation for de-Broglie waves for matter particles

$$\text{Schrödinger Equation : } i\hbar \frac{\partial}{\partial t} \Psi(q,t) = H(P,Q) \Psi(q,t) , \quad Q = q \quad P = -i\hbar \frac{\partial}{\partial q}$$

wave function      classic Hamiltonian      location      differential operator

→ Solution of H-atom: binding energy levels  $E_n$

→ Equivalence to Matrix mechanics

Max Born (1926):

→ Probability interpretation of the wave function  $\Psi(q,t)$ :  $| \Psi(q,t) |^2 dq$  is probability to find in a measurement of the particle's location a value in the interval  $[q, q+dq]$ .  
 $\Psi(q,t)$  is the complex valued probability amplitude.

→  $\Psi(q,t)$  uniquely describes the state of the particle at time  $t$ . This state is in general not associated by a unique position, but a unique probability distribution of the particles position for a measurement of the position at time  $t$ .

Heisenberg's uncertainty principle (1927):

→ Relation links together the standard deviations for the measured values of location and momentum for a particle.

$$\Delta Q \Delta P \geq \frac{\hbar}{2} \quad \Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$$

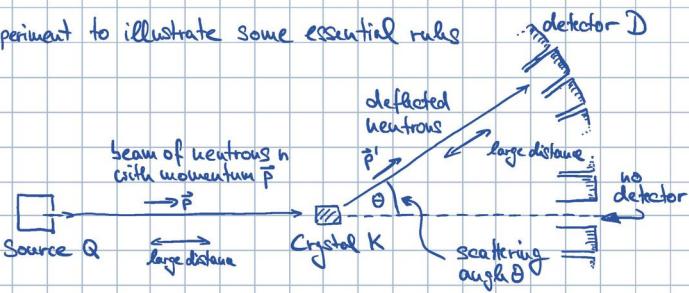
→ Location and momentum cannot be measured arbitrarily precise at the same time.

## 1.2. Main Principles of Quantum Physics

→ We consider a "Gedanken" scattering experiment to illustrate some essential rules of quantum physics

### (A) Basics of a scattering experiment

→ Source Q produces neutrons with fixed momentum  $\vec{p}$  (beam) at large distance from the to be analyzed crystal K



→ Detector D located at large distance from crystal K, detects scattered neutrons in an ideal way:

- does not miss a scattered particle entering D
- can measure  $\vec{p}'$  (which includes also scattering angle  $\theta$ ) very precisely
- the whole solid angle  $4\pi$  is covered by Ds (e.g. whole solid angle covered by detectors)
- except at  $\theta=0$  (no scattering, intensity too high)
- We are only measuring scattering events with  $\theta \neq 0$ !



→ The momentum transfer  $\vec{q} = \vec{p} - \vec{p}'$  is absorbed by the crystal.

We assume the case of elastic scattering off an infinitely heavy scattering center: i.e.

(a) Crystal is infinitely heavy, so that it does not absorb any energy AND can absorb any momentum transfer  $\vec{q}$   
 $\rightarrow |\vec{p}'| = |\vec{p}|$

(b) Scattering process does not modify (e.g. destroy/absorb) any of the atoms in the crystal } assumption given such that one cannot tell (even in principle) at which atom the interaction took place. } up in subsecs F, G

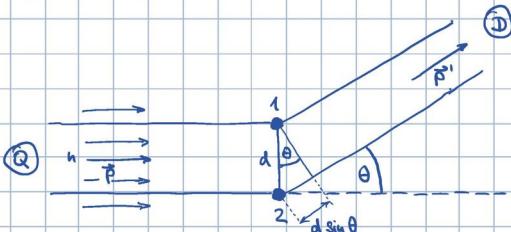
For pedagogical reasons we simplify the setup even further:

(c) Crystal contains 2 foreign atoms (impurities)  $A_1$  and  $A_2$  with distance  $d$  at lattice locations 1 and 2.

Interaction of neutrons with regular crystal atoms negligibly small; only consider scattering off  $A_{1/2}$ .

↪ The resulting setup is equivalent to the double slit experiment.

(d) De Broglie wave length  $\lambda = h/p$  of neutrons  $\Rightarrow$  size of atoms  $A_{1/2}$  (atoms are pointlike)



→ Importance of "large distance": (one standard assumptions for 'scattering')

- (1) The processes in Q (particle production), at the crystal (scattering of crystal atoms) and in D (particle detection and  $p'$  measurement) are independent and not influenced by each other.
- (2) Particles in the beam are all parallel and have exactly the same direction of  $\vec{p}$ .

→ Due makes the following general observations (for any scattering process):

(I) For each incoming neutron which is scattered ( $\theta \neq 0$ ) exactly one detector D detects a scattered particle and measures its momentum  $\vec{p}'$ .

If the particle is not scattered, it goes along the beam leaving the crystal and can in principle be detected as well.

(II) For one single incoming particle we cannot say in which detector the scattered particle will be detected. → In this sense the process is indeterministic.

(III) In a large number of repetitions (with equal initial setup wr. to Q and K) of the scattering process one obtains (within statistical fluctuations) a unique measurement of an angular distribution of the scattered particles → In this sense the process is deterministic.

### (B) 1st Rule of Quantum Physics

→ There is a well-defined and unique probability that a particle emitted from source Q is being observed at a detector D (at  $\theta \neq 0$ )

→ 1st Rule: There exists a complex-valued (probability) amplitude for that to happen which we write as

$\langle D_{\text{fin}} | Q_{\text{in}} \rangle$  ← "Overlap of the state of an incoming particle (produced with  $\vec{p}$  at very early times at Q) with the state of the outgoing particle with  $\vec{p}'$  (being detected at the corresponding detector)."  
 bra/ket notation to be introduced in detail later

Standard notation: In coming : RHS "ket"  
 Out going : LHS "bra" → We will drop subscript "in" and "out" below.

From the amplitude one obtains the probability for the event to happen by taking the modulus squared:

$$\text{Probability} = |\langle D_{\text{fin}} | Q_{\text{in}} \rangle|^2 = \langle D_{\text{fin}} | Q_{\text{in}} \rangle^* \cdot \langle D_{\text{fin}} | Q_{\text{in}} \rangle$$

### (C) 2nd Rule of Quantum Physics



→ We assume that the scattering is a rare process and that multiple scattering of a single particle is negligible

→ The incoming particle has 2 possibilities to get from Q to D

1. From Q to point 1 (with interaction with  $A_1$ ) to D → amplitude  $\langle D | Q \rangle_1$
2. From Q to point 2 (with interaction with  $A_2$ ) to D → amplitude  $\langle D | Q \rangle_2$

We do not have any possibility (accord. to our assumptions) to know which way the particle took.

→ 2nd Rule: IF there is no principle way to know (or measure) which path the particle took to get from Q to D, one has to sum over the amplitudes for all possible processes to obtain the complete amplitude for the process to happen:

$$\langle D | Q \rangle_{1+2} = \underbrace{\langle D | Q \rangle_1}_{=: \phi_{12}} + \underbrace{\langle D | Q \rangle_2}_{=: \phi_1} = \underbrace{\phi_2}_{=: \phi}$$

← If e.g.  $A_2$  would not be there the complete amplitude would be  $\langle D | Q \rangle$

## (D) 3rd Rule of Quantum Physics

→ The process paths related to amplitudes  $\phi_1$  and  $\phi_2$  are related to a sequence of independent processes.

- E.g.  $\phi_1$ :  
 1. Particle propagates from Q to point 1 → amplitude  $K(1, Q)$   
 2. Interaction between particle and  $A_1$ , → amplitude  $W_1$   
 3. Scattered particle propagates from point 1 to D → amplitude  $K(D, 1)$

→ 3rd Rule: IF several independent processes take place within a considered process, one has to multiply the amplitudes of the subprocesses to obtain the complete amplitude.

$$\hookrightarrow \phi_i = \langle D | Q \rangle_i = K(D, i) W_i K(i, Q) \quad i = 1, 2$$

## (E) Interference

→  $\phi_1$  and  $\phi_2$  are complex-valued and we can write  $\phi_1 = |\phi_1| e^{i\varphi_1}$ ,  $\phi_2 = |\phi_2| e^{i\varphi_2}$

Complex phase

→ If e.g. only  $A_1$  would be present,  $\omega_{12} = |\phi_{12}|^2 = |\phi_1 + \phi_2|^2 = |\phi_1|^2 + |\phi_2|^2 + 2|\phi_1||\phi_2| \cos(\varphi_2 - \varphi_1)$  would be the probability that the scattering and the phase  $\varphi_1$  would be irrelevant. However,  $A_1$  and  $A_2$  exist and the probability for the scattering process is

$$\text{probability}(D \leftarrow Q) : \omega_{12} = |\phi_{12}|^2 = |\phi_1 + \phi_2|^2 = \underbrace{|\phi_1|^2 + |\phi_2|^2}_{\omega_1 + \omega_2} + \underbrace{2|\phi_1||\phi_2| \cos(\varphi_2 - \varphi_1)}_{\text{interference term}} \neq \omega_1 + \omega_2 !$$

The probability for a process that can take place via different processes for which there is no principle way to tell/measure which is being realized is in general not the sum of probabilities of the different processes due to possible phase differences.

We determine the interference effect for our scattering example (see figure with arrangement of  $A_{1/2}$ ) assuming that  $A_1$  and  $A_2$  are equal (so  $W_1 = W_2 = W$ ,  $K(1, Q) = K(2, Q) = K(Q)$ )

$$\hookrightarrow \phi_{12} = (K(D, 1) + K(D, 2)) \cdot W \cdot K(Q)$$

 $r_{1/2}$  = distance of  $A_{1/2}$  to D

We will later convince ourselves that  $K(D, 1) = \frac{a}{r_1} e^{ip_1/r_1}$ ,  $K(D, 2) = \frac{a}{r_2} e^{ip_2/r_2}$ ,  $p = |\vec{p}| = |\vec{p}'|$   
 "very" ↓

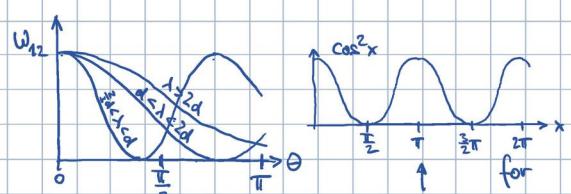
$$\text{Because } r_{1/2} \gg d : r_1 = r_2 \Rightarrow \omega_1 = \omega_2 = \omega \quad \text{AND} \quad r_2 - r_1 = d \sin \theta \Rightarrow \varphi_2 - \varphi_1 = \frac{p(r_2 - r_1)}{\hbar} = \frac{pd \sin \theta}{\hbar}$$

$$\text{and thus } \omega_{12} = 2\omega \left[ 1 + \cos\left(\frac{pd \sin \theta}{\hbar}\right) \right] = 4\omega \cos^2\left(\frac{pd \sin \theta}{\hbar}\right) = 4\omega \cos^2\left(\frac{d\pi \sin \theta}{\lambda}\right) \quad \lambda = \frac{\hbar}{p}$$

→ The smaller  $\lambda$  the more interference minima exist.

There are  $n$  interference minima for

$$\frac{2d}{2n+1} < \lambda < \frac{2d}{2n-1}$$



$n$  minima visible for  $0 < x < \bar{x}$   
 if  $\frac{2n-1}{2}\pi < \bar{x} < \frac{2n+1}{2}\pi$

### Comments:

(1) In a more complete treatment we have to account for the scattering off all atoms in the crystal.

We then have to sum over the location of all lattice positions:

$$\rightarrow \langle D | Q \rangle = \sum_i K(i,i) W_i K(i,Q)$$

(2) Likewise, in a more complete treatment we also have to account for multiple scattering contributions. In this context the previous treatment represents the first term of a series of terms accounting for increasing number of scatterings, i.e.

$$\rightarrow \langle D | Q \rangle = \sum_i K(i,i) W_i K(i,Q) + \sum_{i,j} K(i,i) W_i K(j,i) W_j K(i,Q) + O(W^3)$$

↑ terms of order  $W^3$   
or higher

### (F) Scattering with Spin-Dependence



→ Up to now we have assumed that the neutrons and the crystal atoms do not have any internal structure that might be exchanged in the scattering process.

→ However, neutrons carry and atoms can carry a spin, which is an intrinsic form of angular momentum that is quite similar to the polarization freedom of light / photons.

The total spin of all particles is a conserved quantity (w.r. to the initial state and the final state of the scattering process), but spin can be exchanged between the particles involved in the scattering process ("spin flip").

→ Neutrons can carry 2 spin states :

For our Göttingen experiment we assume that each of the atoms  $A_i$  can carry also 2 spin states:

"spin up":  $n\uparrow$

"spin down":  $n\downarrow$

"spin up":  $A_i\uparrow$

"spin down":  $A_i\downarrow$

If the atoms  $A_i$  would carry no spin ("scalar", "spin-less"), the considerations of the previous sections remain unchanged due to angular momentum conservation up to the fact that there are two different neutron states (instead of a single one). → Neutron spin state cannot flip.

Case 1: The spins of neutron and atoms  $A_{1,2}$  are either all up or all down.

→ Due to angular momentum conservation: no spin flip possible.

$$\Rightarrow \phi_{12} = \langle D, n\uparrow; A_1\downarrow, A_2\downarrow | Q, n\downarrow; A_1\uparrow, A_2\uparrow \rangle \quad \text{or} \quad \phi_{12} = \langle D, n\uparrow; A_1\uparrow, A_2\uparrow | Q, n\downarrow; A_1\downarrow, A_2\downarrow \rangle$$

There are no differences to the previous considerations described in (B) to (E) up to the fact that the subamplitudes in the calculation of  $\phi_{12}$  are different for the up ( $\uparrow$ ) and the down ( $\downarrow$ ) case.

Case 2: The spin of the incoming neutron is down and the spins of  $A_1, 2$  are initially both up.

→ Accounting for angular momentum conservation the following outcomes are possible:

1. The is no spin flip. → Then is no principle way to tell with which atoms the neutron interacted, so we have to add the amplitudes for interaction with  $A_1$  and  $A_2$  → Interference

$$\hookrightarrow \Psi_0 = \langle D, u\downarrow; A_1\uparrow, A_2\uparrow | Q, u\downarrow; A_1\uparrow, A_2\uparrow \rangle = K(D, 1) W_1 K(1, Q) + K(D, 2) W_2 K(2, Q)$$

↑      ↑      ↑      ↑      ↑

Amplitudes for propagation of the neutron:  
we can assume that there is no difference in the  
Ks for spin up ( $u\uparrow$ ) or spin down ( $u\downarrow$ ) neutrons.

↑      ↑      ↑      ↑      ↑      ↓

amplitudes for  
interaction without  
spin flip

2. The spins of the neutron and of  $A_1$  flip. → After the scattering the spin of  $A_1$  has changed, so that is clear that  $A_2$  has not participated in the process.

$$\hookrightarrow \Psi_1 = \langle D, u\uparrow; A_1\downarrow, A_2\uparrow | Q, u\downarrow; A_1\uparrow, A_2\uparrow \rangle = K(D, 1) W'_1 K(1, Q)$$

↑

amplitudes for  
interaction with  
spin flip

3. The spins of the neutron and of  $A_2$  flip. → After the scattering the spin of  $A_2$  has changed, so that is clear that  $A_1$  has not participated in the process.

$$\hookrightarrow \Psi_2 = \langle D, u\uparrow; A_1\uparrow, A_2\downarrow | Q, u\downarrow; A_1\uparrow, A_2\uparrow \rangle = K(D, 2) W'_2 K(2, Q)$$

↑

amplitudes for  
interaction with  
spin flip

→ Outcomes 2 and 3 are physically distinguishable, because the final states w.r.t. to  $A_1$  and  $A_2$  are different – even if one chooses not to measure/observe the spin of  $A_1$  and  $A_2$ .

In general: Processes for which initial or final states are physically distinguishable – even if one decides not to measure/observe the corresponding observables – do never interfere, i.e. their amplitudes shall never be added.

→ One can ask for the probability that the neutron is being detected at  $D$  without specifying (i.e. restricting) the neutrons or the atoms final state spin. Due the has to add the probabilities of outcomes 1-3 incoherently.

$$\hookrightarrow \text{probability} (D, u\downarrow; A_1\uparrow, A_2\uparrow; \text{any spin} \leftarrow Q, u\downarrow; A_1\uparrow, A_2\uparrow) = |\Psi_0|^2 + |\Psi_1|^2 + |\Psi_2|^2$$

(G) Polarization

→ We assume the in the initial state  $A_1$  and  $A_2$  both carry "spin up", but that Q produces neutrons with 50% spin-up and 50% spin-down. We say that in the initial state the atoms  $A_{1,2}$  are 100% polarized spin-up and the the neutrons are unpolarized.

2  $\psi_0, \psi_1, \psi_2$  are the amplitudes for the processes 1-3 for  $n \nu$  in the initial state.

We call  $x_0$  the amplitude for scattering into D for  $n \uparrow$  in the initial state; it is of the interference type because there is no possible spin-flip and thus the amplitudes for scattering off  $A_1$  and  $A_2$  have to be added to get  $x_0$ .

$$\hookrightarrow \text{probability}(D, n; A_1, A_2; \text{any spin} \leftarrow Q, n \text{ (unpolarized)}; A_1 \uparrow, A_2 \uparrow) = \frac{1}{2} (|\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2) + \frac{1}{2} |x_0|^2$$

→ If the neutrons have an arbitrary polarization (e.g. q-polarized spin up) we have

$$\hookrightarrow \text{probability}(D, n; A_1, A_2; \text{any spin} \leftarrow Q, n \text{ (fraction } q \uparrow, \text{ fraction } (1-q) \downarrow); A_1 \uparrow, A_2 \uparrow)$$

$$= (1-q) (|\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2) + q |x_0|^2, \quad (0 \leq q \leq 1)$$

→ A state with non-trivial polarization ( $0 < q < 1$ ) is called a mixed state.

A state that is 100% in a unique quantum mechanical state (100% polarized) is called a pure state.

### 1.3. Units, Scales and Fundamental Constants of Nature

→ In 1.2.(E) we learned that an interference pattern is only visible if the de-Broglie wave length  $\lambda$  of the probing particle is smaller than the distance between the scattering centers:  $\lambda = \frac{h}{p} < d$ .

In general, the resolution one has is limited by the de-Broglie wave length  $\lambda$  of the probing particle.

We see that through the fundamental relation  $h = \lambda p$  there is a unique connection between the apparently different quantities "distance" and "momentum". We can - in fact - consider them as quantities that are dependent (and not independent as we might naturally feel).

→ Frequently used energy unit: electron volt (eV)

$1 \text{ eV}$  = kinetic energy a particle with one electric elementary charge  $e$  (e.g. positron  $e^+$ , proton  $p^+$ ) receives when traversing a potential difference of  $1 \text{ V}$  (volt).

Using the relation  $E = c\sqrt{c^2m^2 + p^2}$  we see that we can conveniently use

→  $1 \frac{\text{eV}}{c}$  as a unit for momentum and  $1 \frac{\text{eV}}{c^2}$  as a unit for mass.

2 examples for masses of particles:  $m_{\text{electron}} = 0.5 \text{ MeV/c}^2$ ,  $m_{\text{proton}} = m_{\text{nucleon}} = 940 \text{ MeV/c}^2$

→ It is possible to use the unit convention  $c = 1$ :

$$[\text{energy}] = [\text{mass}] = [\text{momentum}] = [\text{eV}]$$

$[x] = \text{"unit of } x\text{"}$

$$[\text{time}] = [\text{distance}] = [\text{m}] \quad \text{from } c = 2.998 \cdot 10^8 \frac{\text{m}}{\text{s}} \Rightarrow 1 \text{ s} = 2.998 \cdot 10^8 \frac{\text{m}}{c}$$

→ Using the relation  $h = \lambda p$  which is valid for photons/light one can easily identify the resolution capacity of light of a given energy.

$$\begin{aligned} h &= (1.240 \times 10^{-6} \text{ m}) \cdot (1 \text{ eV}/c) \\ &= (1.240 \times 10^{-10} \text{ m}) \cdot (10^4 \text{ eV}/c) = (1.240 \times 10^{-10} \text{ m}) \cdot (10 \text{ keV}/c) \\ &= (1.240 \times 10^{-15} \text{ m}) \cdot (10^9 \text{ eV}/c) = (1.240 \times 10^{-15} \text{ m}) \cdot (1 \text{ GeV}/c) \\ &= (1.240 \times 10^{-17} \text{ m}) \cdot (10^{12} \text{ eV}/c) = (1.240 \times 10^{-17} \text{ m}) \cdot (100 \text{ GeV}/c) \\ &= (1.240 \times 10^{-18} \text{ m}) \cdot (10^{12} \text{ eV}/c) = (1.240 \times 10^{-18} \text{ m}) \cdot (1 \text{ TeV}/c) \end{aligned}$$

Relativistic particles (i.e. when  $E = c\sqrt{c^2m^2 + p^2} \gg c^2m$ ) have the same resolution capacity as light since their masses can be neglected.

Visible light:  $360 \text{ nm} = 3.6 \cdot 10^{-7} \text{ m} \leq \lambda_n \leq 780 \text{ nm} = 7.8 \cdot 10^{-7} \text{ m}$  has energies  $E_{\text{vis}}$   
 $3.4 \text{ eV} \approx E_{\text{vis}} \approx 1.6 \text{ eV}$

One can use  $\frac{hc}{\text{eV}}$  as a unit for distance

LEP experiment (1989–2000 at CERN, Geneva): electron ( $e^-$ ) – positron ( $e^+$ ) collision 

with  $e^+$  and  $e^-$  each having 100 GeV energy  $\rightarrow E_{\text{total}} = 200 \text{ GeV}$

$\hookrightarrow$  resolution capacity :  $6.2 \cdot 10^{-18} \text{ m}$  ( $\rightarrow$  covers ranges of the strong and weak interactions)

LHC experiment (since 2009): 2 running modes

$$p^+ - p^+ : E_{\text{total}} = 3.5 \text{ TeV} + 3.5 \text{ TeV} = 7 \text{ TeV}$$

$\hookrightarrow$  resolution capacity :  $1.8 \cdot 10^{-19} \text{ m}$  ( $\rightarrow$  covers ranges of the strong and weak interactions and maybe some new physics, i.e. new structures)



$\rightarrow$  We see that the fundamental constants of nature, such as

$$c = 2.998 \cdot 10^8 \frac{\text{m}}{\text{s}} \quad (\text{speed of light})$$

$$\hbar = 6.582 \cdot 10^{-16} \text{ eV} \cdot \text{s} \quad (\text{reduced Planck constant})$$

$$k = 8.617 \cdot 10^{-5} \frac{\text{eV}}{\text{K}} \quad (\text{Boltzmann constant}),$$

intrinsically connect the quantities distance, time, energy, momentum, mass and temperature.

It is thus possible to use units such that  $c = \hbar = k = 1$ . This is the "natural units" convention that is used e.g. in high-energy physics and many other branches of physics.

$$\hookrightarrow [\text{energy}] = [\text{mass}] = [\text{momentum}] = [\text{distance}] = [\text{time}] = [\text{temperature}] = [\text{eV}]$$

$$\hookrightarrow 1 \text{ s} = 1.519 \cdot 10^{15} \text{ eV}^{-1} = (6.582 \cdot 10^{-16} \text{ eV})^{-1}$$

$$1 \text{ m} = 5.068 \cdot 10^6 \text{ eV}^{-1} = (1.973 \cdot 10^7 \text{ eV})^{-1}$$

$$1 \text{ K} = 8.617 \cdot 10^{-5} \text{ eV}$$

One reason why using natural units is useful in physics is that non-trivial functions  $f(x)$  require arguments  $x$  which are dimensionless (i.e.  $x$  must be a number) and that typical functions  $f(x)$  that appear in physics change typically for variations  $\Delta x$  of order unity.