Exercises for T2, Summer term 2016, Sheet 10

1) Pauli matrices

The Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show the following relations:

a) $[\sigma_k, \sigma_l] = 2i\varepsilon_{klm}\sigma_m$ (note: sum convention)

b) $\sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl}\mathbb{1}_2$

c) $\sigma_k\sigma_l = \delta_{kl}\mathbb{1}_2 + i\varepsilon_{klm}\sigma_m$

d) $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbb{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}, \quad \vec{a},\vec{b}\in\mathbb{R}^3$

3) Spin measurements

Initially, let there be a spin-1/2 system in a pure spin state $\chi_{z,+}$, where the spin is pointing in the z-direction.

(a) For which spin operator is $\chi_{z,+}$ an eigenstate and what is the corresponding eigenvalue?

(b) What are the probabilities to get the values $\pm\hbar/2$ if a measurement of the spin in x-direction is taken? What are the corresponding probabilities for a measurement of the spin in y-direction? What are the respective expectation values?

(c) Let's suppose you get the value $-\hbar/2$ for an ideal measurement of the spin in x-direction. This means that the spin is still there and not destroyed after the measurement. In which state is the system after the measurement? Now take another measurement of the spin in the z-direction on this spin. What are the probabilities to get the values $\pm\hbar/2$ for this measurement of the spin in x-direction?

3) Magnetic moment im thermodynamic equilibrium

Let there be a spin-1/2 system in an external magnetic field $\vec{B} = B\vec{e}_3$. The operator of the magnetic moment is $\vec{\mu} = \gamma\vec{S}$ and the Hamilton operator is $H = -\vec{\mu}\cdot\vec{B}$. If the spin is in contact with a heat bath of temperature T, then the corresponding equilibrium state is described by the density matrix

$$\rho = \mathcal{N}\exp(-\beta H), \quad \beta = 1/kT$$

Determine the normalization factor $\mathcal{N}$. Calculate the expectation value and the mean square deviations of μ_i $(i = 1,2,3)$ and H. Sketch the expectation value of μ_3 as function of temperature T.

4) Transformation law for density matrices under spatial rotations

Let $U(\theta\,\vec{e_z})$ be the spin-1/2 rotation matrix that actively rotates a state by an angle θ around the z-axis.

(a) Show:

$$U(\theta\,\vec{e_z})(\vec{n}\cdot\vec{\sigma})U(\theta\,\vec{e_z})^\dagger = \vec{n}'\cdot\vec{\sigma}, \qquad \vec{n}\in\mathbb{R}^3,$$

where $\vec{n}' = R(\theta\,\vec{e_z})\vec{n}$ and $R(\vec{\alpha})$ is the spatial rotation matrix that rotates vectors by an angle $|\vec{\alpha}|$ around the axis $\vec{\alpha}/|\vec{\alpha}|$.

(b) Use the result from (a) to determine the transformation law for the density matrix

$$\rho = \frac{1}{2}(\mathbb{1}_2 + \vec{n}\cdot\vec{\sigma}), \qquad |\vec{n}| = 1.$$

under a rotation given by $U(\theta\,\vec{e_z})$, and argue why $U(\theta\,\vec{e_z})$ indeed has the physical interpretation described above.

Note: In the lecture it was wrongly mentioned that $\vec{n}' = R(-\theta\,\vec{e_z})\vec{n}$. This has been corrected in the online lecture notes.