Exercises for T2, Summer term 2016, Sheet 9

1) Particle scattering at the delta potential

Calculate the energy eigensolution of a particle which scatters at the delta potential from exercise (8.5) and is incoming from the left. Derive this "scattering solution" in ananlogy with the calculations in Chapter 3.3. For this one has to solve the eigenvalue equation of H for positive energies. In addition, calculate the probability for the particle being reflected and the probability that it is transmitted. Sketch these probabilities with respect to their dependence on the energy E.

2) General uncertainty principle

The variance $(\Delta_\omega A)^2$ of some observable A with respect to a state ω is defined by

$$(\Delta_\omega A)^2 = \omega((A - \omega(A))^2)$$

Given two hermitian operators $A, B \in \mathrm{L}(\mathcal{H})$, show that for an arbitrary state ω the inequality

$$\Delta_\omega A \Delta_\omega B \geq |\omega(\frac{i}{2}[A, B])|.$$

holds. For this one can use the (non-hermitian) operator

$$C = \frac{A - \omega(A)}{\Delta_\omega A} + i\frac{B - \omega(B)}{\Delta_\omega B}$$

and the functional properties (a)-(c) of a general state ω as discussed in Chapter 4.2 of the lecture.

3) Mixed state I

Show that a state which is given by the density matrix ρ is a mixed state if $\rho^2 \neq \rho$ holds.

4) Mixed state II

Show that a state which is given by the density matrix ρ is a mixed (pure) state if $\mathrm{Tr}[\rho^2] < 1$ ($\mathrm{Tr}[\rho^2] = 1$) holds.

5) Harmonic oscillator in thermal equilibrium

Given a harmonic osciallator with angular frequency ω which is in thermal equilibrium with an external heatbath of absolute temperature T. The density matrix then has the form:

$$\rho = \frac{\exp(-\mathbf{H}/kT)}{\mathrm{Tr}[\exp(-\mathbf{H}/kT)]},$$

where $\mathbf{H}$ is the Hamilton operator and k the Boltzmann constant.

(a) Calculate the spectral representation of the mixed state ρ in Bra-Ket notation as a function of the temperature, where $|\phi_n\rangle$ is the normalized eigenstate with occupation number n. Note that the sum of the geometric series is very helpful for this calculation.

(b) Calculate the average occupation number $\langle N \rangle$ and the average energy $\langle H \rangle$ as a function of temperature T. ($\langle N \rangle = (\exp(\hbar\omega/kT) - 1)^{-1}$)

(c) Calcuate the average occupation number for visible light ($\lambda = 550$nm) at room temperature ($T = 295$K) and at the surface of the sun ($T = 5500$K).