Exercises for T2, Summer term 2016, Sheet 7

1) Ladder operators

The ladder operators of a one-dimensional harmonic oscillator $a, a^\dagger$ fullfill the commutation relation $[a, a^\dagger] = \mathbb{1}$. Show that:

1. $\left[a, (a^\dagger)^n\right] = n(a^\dagger)^{n-1}$
2. $\left[a, f(a^\dagger)\right] = f'(a^\dagger)$

Assume that the function f is defined as a power series.

2) Coherent state I

The state $|z\rangle \equiv |\psi_z\rangle$ (with $z \in \mathbb{C}$) of a harmonic oscillator is defined via the eigenvalue equation $a|z\rangle = z|z\rangle$. Show that the solution of this equation is

$$|z\rangle = Ce^{za^\dagger}|0\rangle \,.$$

Use the equations derived in excersice (1). The state $|z\rangle$ is an example of a coherent state.

3) Coherent state II

Write the coherent state $|z\rangle$ as a linear combination of the normalized energy eigenstates $|n\rangle \equiv |\phi_n\rangle$ of the harmonic oscillator. Calculate $\langle n|z\rangle$. Determine the normalization constant C (up to an arbitrary phase $e^{i\alpha}$) by imposing the normalization condition $\langle z|z\rangle = 1$.

4) Expectation values for the harmonic oscillator

Calculate the expectation values $\langle X\rangle_n$, $\langle P\rangle_n$, $\langle X^2\rangle_n$, $\langle P^2\rangle_n$ for the energy eigenstates $|n\rangle$ of the harmonic oscillator. Use algebraic methods with the ladder operators.

5) Uniqueness of the eigenstates

Show – under the assumption of a unique ground state – that for every energy eigenvalue of the one-dimensional harmonic oscillator, there exists exactly one corresponding eigenstate. It is sufficient to prove the analogous statement for the number operator N.

6) Completeness

Show that with the energy eigenstates $|n\rangle \equiv |\phi_n\rangle$ one can really identify all eigenstates of the one-dimensional harmonic oscillator. Perform a proof by contradiction, by first assuming there would be an eigenstate $|\nu\rangle$ of the number operator N with the eigenvalue $\nu = n + \alpha$, with $n \in \mathbb{N}$ and $0 < \alpha < 1$. Then use the state $a^{n+1}|\nu\rangle$ to show that this would lead to a contradiction with the general properties of a one-dimensional harmonic oscillator.