Exercises for T2, Summer term 2016, Sheet 5

1) Real valued wavefunction

Consider the **special case** of a **real valued** wavefunction $\varphi(x)$ in one dimension which vanishes for $x \to \pm\infty$. Show that in this case the expectation value of the momentum operator is zero.

2) Consider the wave function of a particle with one degree of freedom of the form

$$\psi(x) = \varphi(x) e^{ip_0 x/\hbar}$$

with p_0 being real and a **real valued** function φ which fulfills:

$$\int_{-\infty}^{+\infty} dx\, \varphi(x)^2 = 1\,,$$

This means $\varphi(x)$ is a function similar to the one given in **1)** Calculate the expectation value of the momentum operator. What is the physical meaning of p_0?

3) Momentum operator

Use the wave function $\psi(x)$ of the Gaussian wave packet from exercise **(4.5)** to calculate the expectation value of P and P^2. Check the uncertainty principle.

4) Calculation in momentum space

Calculate the momentum space wave function $\tilde{\psi}(p)$ of the Gaussian wave packet from exercise **(4.5)**. Furthermore calculate the respective expectation value of X, X^2, P and P^2. Try to be efficient and use e.g. a suitable substitution of variables to get easier integrals or expressions known from earlier exercises.

5) Momentum space wave function with minimal uncertainty

The momentum space wave functions $\tilde{\psi}(p)$ which minimize the product of position and momentum uncertainty $\Delta X \Delta P = \hbar/2$ are characterized by the equation:

$$\left(\frac{X - x_0}{\sigma} + i\, \frac{p - p_0}{\hbar/2\sigma} \right) \tilde{\psi}(p) = 0$$

where p is the momentum operator in momentum space representation and $X = i\hbar\, \partial/\partial p$ the position operator in momentum space representation. x_0 and p_0 are the expectation value of the position and momentum operator respectively. Furthermore the position uncertainty is given by $\sigma = \Delta X$. By solving the differential equation, calculate $\tilde{\psi}(p)$ which fulfills the usual normalization condition

$$\int_{-\infty}^{+\infty} dp\, |\tilde{\psi}(p)|^2 = 1$$

6) Position space wave function with minimal uncertainty

Use the transformation from momentum to position space to calculate the position space wave function $\psi(x)$ which minimizes the product between position and momentum uncertainty $\Delta X \Delta P = \hbar/2$. Start from the momentum space wave function from exercise **(5)**.

7) Commuting operators

Suppose A, B and C are linear operators which fulfill $[A, C] = [B, C] = 0$. Does this also mean that $[A, B] = 0$ holds?

8) Two-dimensional Hilbert space and representation of bra- and ket-vectors

Suppose a two-dimensional complex valued Hilbert space with an orthonormal basis $\{|a_1\rangle, |a_2\rangle\}$ (a-representation). Two vectors are given by:

$$|b_1\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + i\,|a_2\rangle) \quad |b_2\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle - i\,|a_2\rangle)\,.$$

(a) Show that $\{|b_1\rangle, |b_2\rangle\}$ also form an orthonormal basis (b-representation).

(b) Write down the coordinate representation of the ket-vectors $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$, $|b_2\rangle$ and of the respective bra-vectors in the a-representation.

(c) Write down $|a_1\rangle$, $|a_2\rangle$ as functions of $|b_1\rangle$, $|b_2\rangle$ and calculate the corresponding coordinate representation.

(d) Calculate the entries (in terms of scalar products $\langle b_i|a_j\rangle$, $i, j = 1, 2$) of the 2×2 matrix which is transforming a vector from the a- to b-representation. Use the completeness relation from the lecture.

9) Abstract linear operator

Suppose some linear operator T which acts on a complex-valued Hilbert space is defined by $T := |u\rangle\langle u|$ (with $|u\rangle \neq 0$).

(a) Is T hermitian?

(b) What feature is needed from $|u\rangle$ so that T is a projection operator?

(c) Suppose B is an arbitrary linear operator which acts on the same Hilbert space. Show that the trace of the operator TB is given by $\langle u|B|u\rangle$. Recall that the trace of an operator does not depend on the used basis.