Exercises for T2, Summer term 2016, Sheet 4

1) Parametric integral

Calculate the integral that depends on the parameter $u > 0$

$$I(u) = \int_0^\infty dr\, e^{-ur}.$$

Then calculate

$$\int_0^\infty dr\, r^n\, e^{-ur}$$

($n \in \mathbb{N}$) from $I(u)$ without doing any additional integration.

2) H-Atom

The wave function of the ground state of an electron in an hydrogen atom has the form

$$\psi(\vec{x}) = \mathcal{N} \exp(-r/a).$$

Here $r = |\vec{x}|$ is the distance to the nucleus, $a = \hbar/m_e \alpha c$ the Bohr radius, m_e the mass of the electron and $\alpha = e^2/\hbar c \simeq 1/137$ the fine structure constant.

(a) What numerical value do you find for the Bohr radius?

(b) How do you have to choose $\mathcal{N}$, such that the wave function is normalized correctly?

Hint: Be efficient and use results form previous excercises.

3) Expectation value

Calculate the expectation value for r^{-1}, r and r^2 for the state given by the wave function in excercise 2.

4) Gaussian integral

(a) Show that $\int_{-\infty}^{+\infty} dx\, \exp(-ax^2) = \sqrt{\pi/a}$, with $a > 0$.

(b) Calculate $\int_{-\infty}^{+\infty} dx\, x\, \exp(-ax^2)$.

(c) Calculate $\int_{-\infty}^{+\infty} dx\, x^2\, \exp(-ax^2)$.

(d) Calculate $\int_{-\infty}^{+\infty} dx\, x^n\, \exp(-ax^2)$, for an arbitrary natural number n.

5) Gaussian wave packet

Consider a wave function in one spatial dimension, given by

$$\psi(x) = \mathcal{N} \exp(-x^2/4\sigma^2)\,, \qquad (\sigma \in \mathbb{R}^+)$$

(a) What is the normalization constant $\mathcal{N}$?

(b) What is the expectation value for a measurement of the position?

(c) Is the square of the position operator X^2 a hermitian operator? What is the expectation value for a measurement of X^2?

(d) Calculate the expected standard deviation Δx, that one will get in the limit of infinitely many position measuremnts (on identical copies, each of them in the state $\psi(x)$).

6) Spatial translation

The operator T_a acts on a wave function $\psi(x)$ (in one spatial dimension) as

$$(T_a\psi)(x) = \psi(x-a), \quad a \in \mathbb{R}$$

Give an intuitive interpretation of the action of T_a. What is the product T_aT_b? What is $T_a^\dagger$, $T_aT_a^\dagger$ and $T_a^\dagger T_a$? Classify the operator T_a with respect to the properties discussed in the lecture (linearity, unitariy, hermiticity).

7) Group properties

Show that $\{T_a | a \in \mathbb{R}\}$ is an abelian group with respect to the product T_aT_b.

8) Projection operators

(a) Show that $PQ = 0$ is a necessary and sufficient condition that $P + Q$ is a projection operators, if P and Q are projection operators.

(b) Show that $[P_1, P_2] = 0$ is a necessary and sufficient condition that P_1P_2 is a projection operator, if P_1 and P_2 are projection operators.