29. The real space RG calculation for the Ising model in $d = 2$ on a triangular lattice in a nonzero magnetic field leads to the approximate result (see exercise 25)

$$K' = \frac{K}{2}\left[\phi(K+\frac{1}{2}h)+\phi(K-\frac{1}{2}h)\right]^2$$

$$h' = h + \frac{1}{2}log\frac{1+3\,e^{-4K-2h}}{1+3\,e^{-4K+2h}} + 3K\left[\phi(K+\frac{1}{2}h)^2-\phi(K-\frac{1}{2}h)^2\right]$$

where $\phi(x) = \frac{1+e^{-4x}}{1+3\,e^{-4x}}$. Find the 5 fixed points of the renormalization group transformations, linearize around them (study one after the other) and discuss the nature of the fixed points. Perform a numerical simulation.

30. We study now the mapping of quantum Ising models to classical Ising models. We consider first the case of the single spin quantum Ising model

$$\hat{H} = \Gamma\sigma^x$$

and demonstrate that its partition function $Z = Tr\,e^{-\beta\hat{H}}$ is represented as the partition function of the classical 1d Ising chain.

In the spirit of the Feynman path integral formulation write $e^{-\beta\hat{H}} = e^{-\frac{\beta}{N}\hat{H}}e^{-\frac{\beta}{N}\hat{H}}...e^{-\frac{\beta}{N}\hat{H}}$ and insert complete sets of $\mid S_z >$ eigenstates of σ^z_i. Insert additionally also complete sets of $\mid S_x >$ eigenstates of σ^x_i, noting that $< S_x|S_z >= \frac{1}{\sqrt{2}}e^{i\pi\,\frac{1-S_x}{2}\,\frac{1-S_z}{2}}$. Verify that $< S'_z|e^{\frac{\beta}{N}\Gamma\sigma^x}|S_z >= c\,e^{J_\tau S'_z S_z}$. Here $J_\tau = artanh(e^{-2\,\delta\tau\,\Gamma})$, $\delta\tau = \frac{\beta}{N}$ and c is an unimportant constant. Demonstrate the equivalence of the single spin quantum Ising model to the Ising chain.

31. Generalize the above arguments to the quantum Ising chain

$$\hat{H} = -J\sum_i \sigma^z_i\sigma^z_{i+1} - \Gamma\sum_i \sigma^x_i$$

showing that it is mapped to the 2d classical Ising model of size $\beta = \frac{1}{T}$ in the imaginary time direction. In the limit $T \to 0$ the 2d classical Ising model has infinite spatial and temporal directions. The coupling constants in the classical model are anisotropic and depend on the discretization step $\delta\tau = \frac{\beta}{N}$

$$J_{spat} = J\,\delta\tau, \qquad J_\tau = arctanh(e^{-2\,\delta\tau\,\Gamma})$$