26. Consider the quantum Ising chain

$$\hat{H} = -J \sum_i \sigma_i^z \sigma_{i+1}^z - \Gamma \sum_i \sigma_i^x$$

where $J > 0, \Gamma \geq 0$. Note that the Pauli matrices commute on different lattice sites, whereas on the same site they anticommute (if they are different) or square to **1** (if they are equal). To investigate the duality symmetry of the quantum Ising chain associate a dual lattice to the original lattice (chain) such that sites in the original chain are associated with the bonds of the dual chain and vice versa. The operators on the dual lattice are defined as

$$\tau_j^x = \sigma_j^z \sigma_{j+1}^z, \qquad \tau_j^z = \prod_{k \leq j} \sigma_k^x$$

This mapping represents the duality transformation. Check that these operators on the dual lattice satisfy the same set of (anti)commutation rules as the Pauli matrices. Rewrite the original quantum ising chain Hamiltonian in terms of the dual operators. For which special value of the coupling constants do you find self duality of the model? Hence, in the case of there being a unique critical point, it would be located at this distinguished value (compare with exercise 22 f).

27. The critical exponents could conceivably be different above and below the critical temperature. Here we will study $\nu(T > T_c)$ and $\nu'(T < T_c)$, and show that they are equal. Start from the scaling relation of the singular part of the free energy (explain this specific formulation)

$$f_s(t,h) = \mid t \mid^{d\bar{\nu}} \phi_{\pm}(\frac{h}{\mid t \mid^{y_h \bar{\nu}}})$$

with $\bar{\nu} = \nu(T > T_c)$ and $\bar{\nu} = \nu'(T < T_c)$. For fixed $h \neq 0$, $f_s(t,h)$ should be a smooth function of t, because the only singularity which we expect is at $t = h = 0$. Show that $f_s(t,h)$ can be written in the form

$$f_s(t,h) = h^{\frac{d}{y_h}} \chi_{\pm}(\frac{h}{\mid t \mid^{y_h \bar{\nu}}})$$

and explain how the smoothness assumption mentioned above constrains the analytic form of the functions $\chi_{\pm}$. Hence show that $v = v'$.

28. Suppose that a specific d-dimensional lattice model for the critical behaviour of spins leads to renormalization group equations of the form

$$p' = b^2 p + c_1(b^2 - b^{\epsilon})q - c_2 pq$$
$$q' = b^{\epsilon} q$$

Here p, q are coupling constants, c_1, c_2 are positive real constants, $\epsilon = 4 - d$, b is the usual length scaling factor. Find the fixpoint of the renormalization group transformation, linearize around it and determine the corresponding matrix of the linearized flow. Find the associated scaling variables and their renormalization group eigenvalues. Discuss the nature of the fixed point.