13. Consider the microcanonical ensemble for a system of N spins with Hamiltonian $-\mathcal{H} = H \sum_{i=1}^{N} S_i$ where $S_i = \pm 1$. Calculate the number $\Omega(E) = \sum_{\{S_i=\pm 1\}} \delta(E + H \sum_{i=1}^{N} S_i)$ of states with energy E to leading order in N, as well as the corresponding entropy $S(E)$ and temperature T.
Hints: Use $\delta(x) = \int \frac{dk}{2\pi} e^{ikx}$ and perform the integrations with the method of steepest descent.

14. The $1d$-Ising model has the Hamiltonian $-H_\Omega = H \sum_{i=1}^{N} S_i + J \sum_{i=1}^{N} S_i S_{i+1}$ and we consider free boundary conditions. N is the total number of spins, we denote by $N_\pm$ the number of sites with spin ± 1. We define the number D of domains in a given configuration of the chain as the number of maximally connected pieces of spins of the same value in the chain. Rewrite the energy as $-H_\Omega = (J + H)N_+ + (J - H)N_- - 2JD + J$.

15. We study some aspects of the microcanonical ensemble for the $1d$-Ising model with free boundary conditions. By keeping constant the values of N, N_+, D we would like to enumerate the degeneracy of a level with energy E, which ultimately would lead us to the microcanonical ensemble solution (a goal far too complicated for just an exercise of this course). Consider here $D = 2k$, $k \in \mathbf{N}$, $N_+ \geq k$, $N_- = N - N_+ \geq k$. We want to solve the combinatorial problem of how many distinct configurations exist, under fixed values of N, N_+, D.

    Investigate explicitely all cases for $N = 5$ with an even number of domains. Show that generally the combinatorial problem reduces to analyzing the number of different solutions for the following system of equations for nonnegative integer variables u_i, d_i

$$u_1 + ... + u_k = N_+ - k$$

$$d_1 + ... + d_k = N - N_+ - k$$

    Here $u_i \in N \cup \{0\}$, $i = 1, ...k$ represents the (number -1) of $+1$ spins in the $i-th$ of the k domains of $+1$ spins; similarly d_i represents the (number-1) of -1 spins in the $i - th$ of the k domains of -1 spins. Verify from the theory of the composition of an integer into non-negative parts that the number of different solutions obtains as

$$2 \begin{pmatrix} N_+ - 1 \\ k - 1 \end{pmatrix} \begin{pmatrix} N - N_+ - 1 \\ k - 1 \end{pmatrix} \Theta(N_+ - k)\Theta(N - N_+ - k).$$

16. Verify that - apart of a factor of $\frac{1}{2}$ - the previous number of solutions sums up to the total number 2^N of possible configuration for a chain of size N.
Hint: First perform the sum over the allowed values of N_+, then sum over the number of domains.