7. For the $d = 1$ Ising model construct the matrix S which diagonalizes the transfer matrix T. You will find it helpful to write down the matrix elements in terms of the variable ϕ given by $cot(2\phi) = e^{2K} sinh(h)$. As an application evaluate explicitely the two point correlation function $G(i, i + j)$.

8. Reformulate the transfer matrix method for the $d = 1$ Ising model in the case of free boundary conditions and calculate the partition function.
   Hint: You will need to introduce a new matrix in addition to T.

   (a) Convince yourself that in the case of a vanishing magnetic field the partition function is agreeing with the result already obtained in the lecture with other methods.

   (b) Show that in the thermodynamic limit the free energy per spin coincides with the value already obtained in the lecture for periodic boundary conditions.

9. For the $d = 1$ Ising model calculate the magnetization $M = -\frac{\partial f}{\partial H}$ as well as the isothermal magnetic suceptability $\chi_T = \frac{\partial M}{\partial H}$ from the solution of the free energy per spin f in the thermodynamic limit. Perform plots of M versus h for various temperatures T as well as of χ_T versus T for various values of the magnetic field.

10. Show that the isothermal magnetic suceptability χ_T can also be calculated from the two point correlation function $G(i, i + j)$ via $\chi_T = \beta \sum_j G(i, i + j)$. By inserting the explicit result for $G(i, i + j)$ verify that χ_T calculated in this way agrees with the value obtained in exercise 9.
    Note: In the thermodynamic limit the sum over j runs from $-\infty$ to $+\infty$.

11. Generalise the transfer matrix formalism to the $d = 2$ Ising model: Suppose that there are N rows parallel to the x axis and M rows parallel to the y axis. We will require $N \to \infty$, while considering $M = 1$ or $M = 2$ only. Periodic boundary conditions are chosen in both directions. The magnetic field is set to be zero.

$$-\beta H_\Omega = K \sum_{n=1}^{N} \sum_{m=1}^{M} (S_{m,n} S_{m+1,n} + S_{m,n} S_{m,n+1}$$

    For the case $M = 1$ show that the transfer matrix is a 2×2 matrix, and show that its eigenvalues are

$$\lambda_1 = 1 + x^2, \quad \lambda_2 = x^2 - 1$$

    where $x = e^K$.

12. Now consider the case $M = 2$. We need to extend the transfer matrix formalism. Consider the vector

$$v_n = (S_{1,n}, S_{2,n}, ..., S_{m,n})$$

which gives the configuration of a row n. Show that

$$H_\Omega = \sum_{n=1}^{N} E_1(v_n, v_{n+1}) + E_2(v_n)$$

where E_1 is the energy of interaction between neighbouring rows and E_2 is the energy of a single row. Hence show that

$$Z = \sum_{v_1, v_2, ... v_n} T_{v_1 v_2} T_{v_2 v_3} ... T_{v_N v_1}$$

where T is a transfer matrix of dimensions $2^M \times 2^M$ whose form you should give. Calculate T and find its eigenvalues.