Exercises for nonlocality, entanglement und geometry of quantum systems
Sheet 7

Prof. Reinhold A. Bertlmann & Philipp Köhler

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Exercise 23

The Werner state is given by

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{1}{4}\mathbb{1}_4$$

In order to check if the state is entangled for certain p, one has to use a separability criterion. A very ease to calculate criterion is the PPT-criterion (positive partial transposition). Calculate the value of p for which the state becomes separable.

PPT-Criterion
A state ρ acting on $\mathcal{H}^2 \otimes \mathcal{H}^2$ is separable if and only if its partial transposition is a positive operator (all eigenvalues are positive),

$$\rho^{T_B} = (\mathbb{1} \otimes T)\rho \geq 0.$$

It does the following to a matrix:

$$\rho_{ij,kl} = \langle i \otimes k|\rho|j \otimes l\rangle = \begin{pmatrix} \rho_{11,\mathrm{kl}} & \rho_{12,\mathrm{kl}} \\ \rho_{21,\mathrm{kl}} & \rho_{22,\mathrm{kl}} \end{pmatrix}$$

$$\text{PPT} \longrightarrow (\mathbb{1} \otimes T)\rho_{ij,kl} = \rho_{ij,lk} = \begin{pmatrix} \rho_{11,\mathrm{lk}} & \rho_{12,\mathrm{lk}} \\ \rho_{21,\mathrm{lk}} & \rho_{22,\mathrm{lk}} \end{pmatrix}$$

That means the PPT criterion applies the following changes to a density matrix:

$$\rho^{T_B} = \begin{pmatrix} \rho_{11,11} & \rho_{11,12} & \rho_{12,11} & \rho_{12,12} \\ \rho_{11,21} & \rho_{11,22} & \rho_{12,21} & \rho_{12,22} \\ \rho_{21,11} & \rho_{21,12} & \rho_{22,11} & \rho_{22,12} \\ \rho_{21,21} & \rho_{21,22} & \rho_{22,21} & \rho_{22,22} \end{pmatrix}$$

Now one has to check if the eigenvalues of this new density matrix are positive or not.

Exercise 24

Write down the Bloch decomposition of the Werner state, as given in Exercise 23.

Exercise 25

One can calculate the violation of a Bell inequality from the coefficient matrix t_{ij} of the Bloch decomposition of a density matrix using a theorem introduced by the Horodecki family. Use this theorem to calculate the violation of the CHSH inequality for the Werner state.

Horodecki Theorem
A very often used form of the Bell inequality is the CHSH form, which was introduced by Clauser, Horne, Shimony and Holt.

$$-2 \leq E(\vec{a},\vec{b}) + E(\vec{a}',\vec{b}) + E(\vec{a},\vec{b}') - E(\vec{a}',\vec{b}') \leq 2$$

Every density matrix can be decomposed into a Bloch basis.

$$\rho = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \vec{r}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{u}\cdot\vec{\sigma} + \sum_{i,j=1}^{3} t_{ij}\sigma_i\otimes\sigma_j\right)$$

The violation of a Bell inequality can be calculated from the coefficient matrix of the Bloch decomposition $T = t_{ij}$ by calculating $T^T T$ and taking the two bigger eigenvalues of this matrix. The violation for a Bell inequality then amounts to

$$\mathcal{B} = 2\sqrt{u + u'} > 2$$

where u, u' are the Eigenvalues of $T^T T$.