38. $\mathrm{Tr}\,[\hat{x}, \hat{p}] = 0$, weil $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$

Aber f. Operatoren im Hilbertraum ist i.a., insbesondere f. $\hat{x}$ u. $\hat{p}$, die Spur nicht definiert!

39. a) Reflexion rechts: $|S_{RR}|^2$

" links: $|S_{LL}|^2$

Transmission nach rechts: $|S_{LR}|^2$

" " links: $|S_{RL}|^2$

Absorption rechts: $1 - |S_{RR}|^2 - |S_{RL}|^2$

" links: $1 - |S_{LL}|^2 - |S_{LR}|^2$

b) $\{|X, \text{ein}\rangle\}$ und $\{|Y, \text{aus}\rangle\}$ sind ONB

und $|Y, \text{aus}\rangle = \sum_X S_{XY} |X, \text{ein}\rangle$

$\Rightarrow$ S beschreibt den Basiswechsel

$\Rightarrow$ S erhält das innere Produkt $\Rightarrow$ S <u>unitär</u>

40. a) S unitär $\Rightarrow$ $S^\dagger = S^{-1}$ $\Rightarrow$ $\det S = \frac{1}{(\det S)^*}$ $\Rightarrow \det S = e^{i\delta}$

$$\Rightarrow \begin{pmatrix} s_{11}^* & s_{21}^* \\ s_{12}^* & s_{22}^* \end{pmatrix} = e^{-i\delta} \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix}$$

$$\Rightarrow s_{22} = e^{i\delta} s_{11}^*, \quad s_{21} = -e^{i\delta} s_{12}^*$$

Setze $r = e^{-i\frac{\delta}{2}} s_{11}$, $t = e^{-i\frac{\delta}{2}} s_{12}$ $\Rightarrow$ $S = e^{i\frac{\delta}{2}} \begin{pmatrix} r & t \\ -t^* & r^* \end{pmatrix}$

$\det S = e^{i\delta} \Rightarrow |r|^2 + |t|^2 = 1$

Ersetze z.B. $|X, \text{aus}\rangle$ durch $e^{i\frac{\delta}{2}} |X, \text{aus}\rangle$ (trivialer Basiswechsel)

$\Rightarrow \delta \to 0$

40.b) $\delta_R = \arg S_{RR} - \arg S_{RL}$

$\delta_L = \arg S_{LL} - \arg S_{LR}$

$\delta_L + \delta_R = \arg r^* - \arg(-t^*) + \arg r - \arg t$

$= \arg |r|^2 - \arg(-|t|^2) = -\pi = \pi \mod 2\pi$

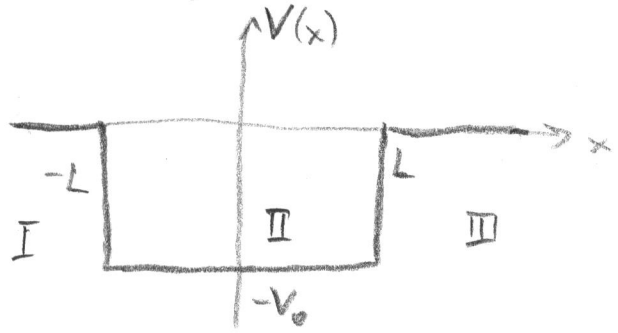
41.a) S invariant unter $L \leftrightarrow R$, [illegible] $\Rightarrow r^* = r,\ t^* = -t$

$\Rightarrow S = \begin{pmatrix} r & t \\ t & r \end{pmatrix} \Rightarrow \delta_L = \delta_R = \pm\frac{\pi}{2}$

b) $|r|^2 = |t|^2 = \frac{1}{2},\ r^* = r,\ t^* = -t \Rightarrow$

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 & \pm i \\ \pm i & \pm 1 \end{pmatrix}$$

42. Vorlesung:



Gebundene Zustände

$E = E_n < 0, \quad n = 1, 2, \ldots$

a) n ungerade

$$\varphi_n = \varphi^{(+)}(x) = \begin{cases} B e^{kx} & \text{I} \\ D \cos qx & \text{II} \\ B e^{-kx} & \text{III} \end{cases} \qquad k \equiv \frac{1}{\hbar}\sqrt{-2mE} \qquad q \equiv \frac{1}{\hbar}\sqrt{2m(E+V_0)}$$

Stetig in $x = \pm L \Rightarrow D = B \dfrac{e^{-kL}}{\cos qL}$

Normierung:

$$\int_{-\infty}^{\infty} |\varphi^{(+)}|^2 dx = 2B^2 \frac{e^{-2kL}}{2k} + D^2 \int_{-L}^{L} \cos^2 qx \, dx$$

$$\int_{-L}^{L} \cos^2 qx \, dx = \frac{1}{2}\int_{-L}^{L} (1 + \cos 2qx) dx = \frac{1}{2}\left(2L + 2\frac{\sin 2qL}{2q}\right)$$

$$= L + \frac{\sin qL \cos qL}{q}$$

$$\Rightarrow \int_{-\infty}^{\infty} |\psi^{(+)}|^2 dx = B^2 \left[\frac{e^{-2\kappa L}}{\kappa} + \frac{e^{-2\kappa L}}{\cos^2 qL} \left(L + \frac{\sin qL \cos qL}{q} \right) \right]$$

$$= B^2 e^{-2\kappa L} \left[\frac{1}{\kappa} \left(1 + \underbrace{\frac{\kappa}{q} \operatorname{tg} qL}_{} \right) + \frac{L}{\cos^2 qL} \right]$$

Vorlesung: $\frac{\kappa}{q} = \operatorname{tg} qL \Rightarrow \operatorname{tg}^2 qL$

$$= B^2 \frac{e^{-2\kappa L}}{\kappa \cos^2 qL} (1 + \kappa L) \stackrel{!}{=} 1$$

$$\Rightarrow B_n = \sqrt{\kappa_n} \frac{\cos q_n L}{\sqrt{1 + \kappa_n L}} e^{\kappa_n L}$$

f. n ungerade

$$\Rightarrow D_n = \sqrt{\frac{\kappa_n}{1 + \kappa_n L}}$$

b) n gerade

$$\psi_n = \psi^{(-)}(x) = \begin{cases} -Be^{\kappa x} & \text{I} \\ C \sin qx & \text{II} \\ Be^{-\kappa x} & \text{III} \end{cases}$$

Stetig in $x = \pm L \Rightarrow C = B \frac{e^{-\kappa L}}{\sin qL}$

Normierung:

$$\int_{-\infty}^{\infty} |\psi^{(-)}|^2 dx = 2B^2 \frac{e^{-2\kappa L}}{2\kappa} + C^2 \int_{-L}^{L} \sin^2 qx \, dx$$

$$\int_{-L}^{L} \sin^2 qx = \frac{1}{2}\int_{-L}^{L}(1-\cos 2qx)\,dx = \frac{1}{2}\left(2L - 2\,\frac{\sin 2qL}{2q}\right)$$

$$= L - \frac{\sin qL \cos qL}{q}$$

$$\Rightarrow \int_{-\infty}^{\infty} |\varphi^{(-)}|^2 dx = B^2\left[\frac{e^{-2\kappa L}}{\kappa} + \frac{e^{-2\kappa L}}{\sin^2 qL}\left(L - \frac{\sin qL\cos qL}{q}\right)\right]$$

$$= B^2 e^{-2\kappa L}\left[\frac{1}{\kappa}\left(1 - \underbrace{\frac{\kappa}{q}\operatorname{ctg} qL}\right) + \frac{L}{\sin^2 qL}\right]$$

Vorlesung: $\frac{\kappa}{q} = -\operatorname{ctg} qL \Rightarrow \operatorname{ctg}^2 qL$

$$= B^2 \frac{e^{-2\kappa L}}{\kappa \sin^2 qL}\,(1+\kappa L) \overset{!}{\doteq} 1$$

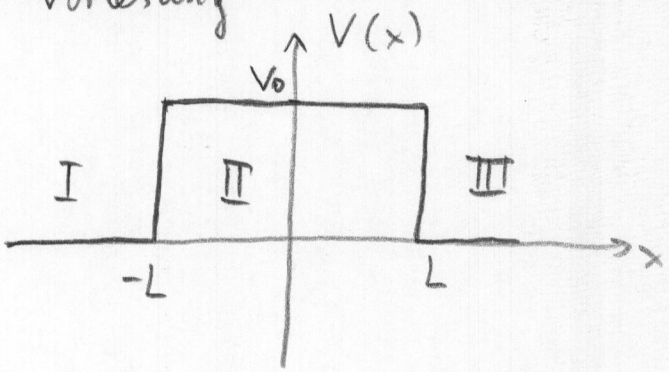
$$\Rightarrow B_n = \sqrt{\kappa_n}\,\frac{\sin q_n L}{\sqrt{1+\kappa_n L}}\,e^{\kappa_n L}$$

f. n gerade

$$\Rightarrow C_n = \sqrt{\frac{\kappa_n}{1+\kappa_n L}}$$

d.h. im Resultat f. n ungerade ist einfach $\cos q_n L$ durch $\sin q_n L$ zu ersetzen, wie zu erwarten

43. Vorlesung:



$0 < E < V_0$

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{I} \\ \tilde{C}e^{-qx} + \tilde{D}e^{qx} & \text{II} \\ Fe^{ikx} & \text{III} \end{cases}$$

$$k \equiv \frac{1}{\hbar}\sqrt{2mE}$$

$$q \equiv \frac{1}{\hbar}\sqrt{2m(V_0 - E)}$$

Stetigkeit von ψ u. ψ' =>

$$\psi(-L) = Ae^{-ikL} + Be^{ikL} = \tilde{C}e^{qL} + \tilde{D}e^{-qL} \qquad \text{(i)}$$

$$\psi'(-L) = ik(Ae^{-ikL} - Be^{ikL}) = q(-\tilde{C}e^{qL} + \tilde{D}e^{-qL}) \qquad \text{(ii)}$$

$$\psi(L) = \tilde{C}e^{-qL} + \tilde{D}e^{qL} = Fe^{ikL} \qquad \text{(iii)}$$

$$\psi'(L) = q(-\tilde{C}e^{-qL} + \tilde{D}e^{qL}) = ikFe^{ikL} \qquad \text{(iv)}$$

$$\text{(iii)} \cdot q + \text{(iv)} \Rightarrow \tilde{D} = \frac{q + ik}{2q}Fe^{ikL}$$

$$\text{(iii)} \cdot q - \text{(iv)} \Rightarrow \tilde{C} = \frac{q - ik}{2q}Fe^{ikL}$$

Einsetzen in (i) u. (ii):

$$(i) \rightarrow A e^{-ikL} + B e^{ikL} = \left(\frac{q-ik}{2q} e^{2qL} + \frac{q+ik}{2q} e^{-2qL} \right) F e^{ikL}$$

$$\Rightarrow A e^{-2ikL} + B = \left(\cosh 2qL - i \frac{k}{q} \sinh 2qL \right) F \quad (I)$$

$$(ii) \rightarrow ik \left(A e^{-ikL} - B e^{ikL} \right) = \left(\frac{ik-q}{2} e^{2qL} + \frac{q+ik}{2} e^{-2qL} \right) F e^{ikL}$$

$$\Rightarrow A e^{-2ikL} - B = \left(\cosh 2qL + i \frac{q}{k} \sinh 2qL \right) F \quad (II)$$

$$(I)-(II) \rightarrow 2B = -i \left(\frac{k}{q} + \frac{q}{k} \right) \sinh(2qL) \cdot F$$

$$\Rightarrow \frac{B}{F} = -i \frac{k^2+q^2}{2kq} \sinh 2qL$$

$$(I)+(II) \rightarrow 2A e^{-2ikL} = \left[2 \cosh 2qL + i \left(\frac{q}{k} - \frac{k}{q} \right) \sinh 2qL \right] F$$

$$\Rightarrow \underline{\frac{F}{A} = e^{-2ikL} \left[\cosh 2qL + i \frac{q^2-k^2}{2kq} \sinh 2qL \right]^{-1}}$$

(Transmissionsamplitude)

$$\Rightarrow \underline{\frac{B}{A}} = \frac{B}{F} \cdot \frac{F}{A} = -i e^{-2ikL} \frac{(k^2+q^2) \sinh 2qL}{2kq \cosh 2qL + i (q^2-k^2) \sinh 2qL}$$

(Reflexionsamplitude)

$$\Rightarrow T(E) \equiv \frac{|F|^2}{|A|^2} = \left[1 + \frac{(k^2+q^2)^2}{4k^2q^2} \sinh^2 2qL \right]^{-1} \text{ wie in Vorlesung}$$

$$R(E) \equiv \frac{|B|^2}{|A|^2} = \left(\frac{k^2+q^2}{2kq} \right)^2 \sinh^2 2qL \left[1 + \frac{(k^2+q^2)^2}{4k^2q^2} \sinh^2 2qL \right]^{-1}$$

$$\Rightarrow R(E) + T(E) = 1$$

44. $\langle x \rangle_n \equiv \langle n|x|n\rangle \propto \langle n|a+a^\dagger|n\rangle \propto \underbrace{\langle n+1|n\rangle}_{0} + \underbrace{\langle n|n+1\rangle}_{0} = 0$

$$\langle x^2\rangle_n \equiv \langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\langle \underbrace{n}_{\propto\langle n+2|}|a^2 + aa^\dagger + a^\dagger a + \underbrace{a^{\dagger 2}}_{\propto|n+2\rangle}|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|2N+1|n\rangle = \frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right) \equiv x_0^2\left(n+\tfrac{1}{2}\right)$$

$$\Rightarrow \underline{(\Delta x)_n^2} = \langle x^2\rangle_n - \langle x\rangle_n^2 = \underline{x_0^2\left(n+\tfrac{1}{2}\right)}$$

$$\langle p\rangle_n \equiv \langle n|p|n\rangle \propto \langle n|a-a^\dagger|n\rangle \propto \underbrace{\langle n+1|n\rangle}_{0} - \underbrace{\langle n|n+1\rangle}_{0} = 0$$

$$\langle p^2\rangle_n \equiv \langle n|p^2|n\rangle = -\frac{\hbar m\omega}{2}\langle \underbrace{n}_{\propto\langle n+2|}|a^2 - aa^\dagger - a^\dagger a + \underbrace{a^{\dagger 2}}_{\propto|n+2\rangle}|n\rangle$$

$$= \frac{\hbar m\omega}{2}\langle n|2N+1|n\rangle = \frac{\hbar^2}{x_0^2}\left(n+\tfrac{1}{2}\right)$$

$$\Rightarrow \underline{(\Delta p)_n^2} = \langle p^2\rangle_n - \langle p\rangle_n^2 = \underline{\frac{\hbar^2}{x_0^2}\left(n+\tfrac{1}{2}\right)}$$

$$\Rightarrow (\Delta x)_n(\Delta p)_n = \hbar\left(n+\tfrac{1}{2}\right)$$

45. $\langle m|a|n\rangle = \langle m|\sqrt{n}|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$

$$n \cong \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow n \qquad \Rightarrow$$

$$(a_{mn}) \cong \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 \cdots \\ \vdots & & & \sqrt{3} \\ \vdots & & & \ddots \\ & & & \end{pmatrix}$$

$$\langle m|a^+|n\rangle = \langle m|\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$$

$$\Rightarrow (a^+_{mn}) \cong \begin{pmatrix} 0 & 0 & \cdots \\ \sqrt{1} & 0 & \cdots \\ 0 & \sqrt{2} & \ddots \\ & & \end{pmatrix} = (a_{mn})^+$$

46. a) $|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle \underbrace{\langle n|\alpha\rangle}_{\langle 0|\frac{a^n}{\sqrt{n!}}|\alpha\rangle} = \langle 0|\alpha\rangle \sum \frac{\alpha^n}{\sqrt{n!}}|n\rangle$

Normierung: $1 = \langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \underbrace{\sum \frac{|\alpha|^{2n}}{n!}}_{e^{|\alpha|^2}}$

$$\Rightarrow |\langle 0|\alpha\rangle| = e^{-|\alpha|^2/2}$$

$$\Rightarrow |\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad \text{(bis auf Phasenfaktor)}$$

46.b) $\langle N \rangle = \langle \alpha | a^+ a | \alpha \rangle = |\alpha|^2$

$$\underline{P_n} = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = \underline{\frac{\langle N \rangle^n}{n!} e^{-\langle N \rangle}}$$

: Poissonverteilung mit Mittelwert $\langle N \rangle$