24. Cancellation of IR-divergences

at one-loop, we had

$$F_1(q^2) = \underset{\text{tree-contribution}}{1} + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$

$$\left\{ \ln \frac{(1-x)^2 m^2}{(1-x)^2 m^2 - q^2 yz} + \frac{m^2(1-4x+x^2)+q^2(1-y)(1-z)}{(1-x)^2 m^2 - q^2 yz + x m_\gamma^2} \right.$$

$$\left. - \frac{m^2(1-4x+x^2)}{(1-x)^2 m^2 + x m_\gamma^2} \right\}$$

contains IR-divergent contributions

(in this parametrization for $x \to 1$, $y, z \to 0$)

→ expression for $\frac{d\sigma(p\to p')}{d\Omega}$ is IR-div.

→ cannot correspond to an observable quantity

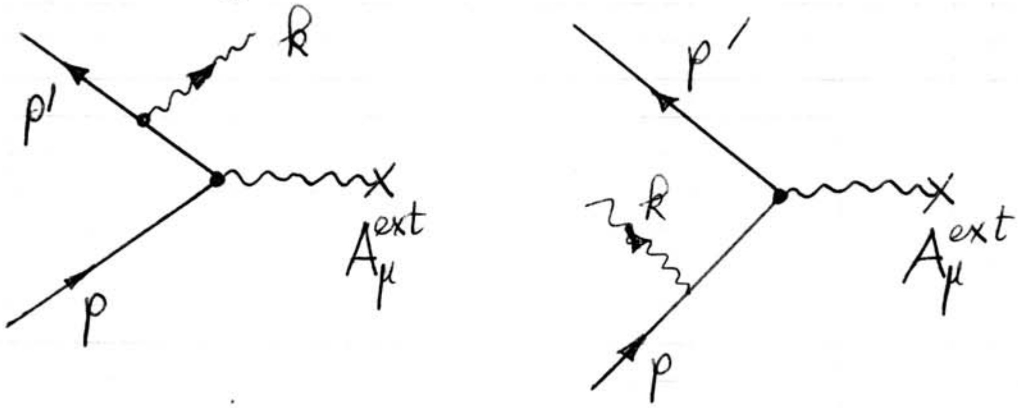
but $d\sigma(p\to p') + d\sigma(p \to p' + \text{soft photons})$ is observable and IR-finite counting powers of e:

$$d\sigma(p\to p') = \underset{\substack{\uparrow \\ \text{tree}}}{e^2 A} \left(1 + \underset{\substack{\uparrow \\ \text{one-loop}}}{e^2 B} + \mathcal{O}(e^4)\right)$$

→ at one-loop one can restrict the computation of $d\sigma(p\to p' + \text{photons})$ to the emission of <u>one</u> (soft) photon, as

$$d\sigma(p\to p' + \text{photons}) = e^4 C + \mathcal{O}(e^6)$$

→ corresponds to the contribution of the tree-diagrams



→ IR-divergences in the combination $AB + C$ has to <u>vanish</u>

to see this, we compute the IR-divergent part of $F_1(q^2)$

relevant parameter region $x \approx 1$, $y, z \approx 0$ $\rightarrow$

in numerators: $x = 1$, $y = z = 0$

in denominators: $x m_\gamma^2 \rightarrow m_\gamma^2$

$$\rightarrow F_1(q^2) = 1 + \frac{\alpha}{2\pi} \int_0^1 dx \int_0^{1-x} dy$$

$$\times \left\{ \frac{-2m^2 + q^2}{(1-x)^2 m^2 - q^2 y(1-y-x) + m_\gamma^2} - \frac{-2m^2}{(1-x)^2 m^2 + m_\gamma^2} \right\}$$

$+$ IR-finite terms

variable transformation $y = (1-x)\xi$, $w = 1-x$:

$$\rightarrow F_1(q^2) = 1 + \frac{\alpha}{2\pi} \int_0^1 dw\, w \int_0^1 d\xi \left\{ \frac{-2m^2 + q^2}{w^2 m^2 - q^2 w^2 \xi(1-\xi) + m_\gamma^2} - \frac{-2m^2}{m^2 w^2 + m_\gamma^2} \right\} + \dots$$

$$= 1 + \frac{\alpha}{4\pi} \int_0^1 d\xi \left\{ \frac{-2m^2 + q^2}{m^2 - q^2 \xi(1-\xi)} \ln \frac{m^2 - q^2 \xi(1-\xi)}{m_\gamma^2} + 2 \ln \frac{m^2}{m_\gamma^2} \right\}$$

$$+ \dots$$

$$= 1 + \frac{\alpha}{2\pi} \ln m_\gamma^2 \Big[\underbrace{\int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - q^2 \xi(1-\xi)} - 1}_{=: f_{IR}(q^2)} \Big]$$

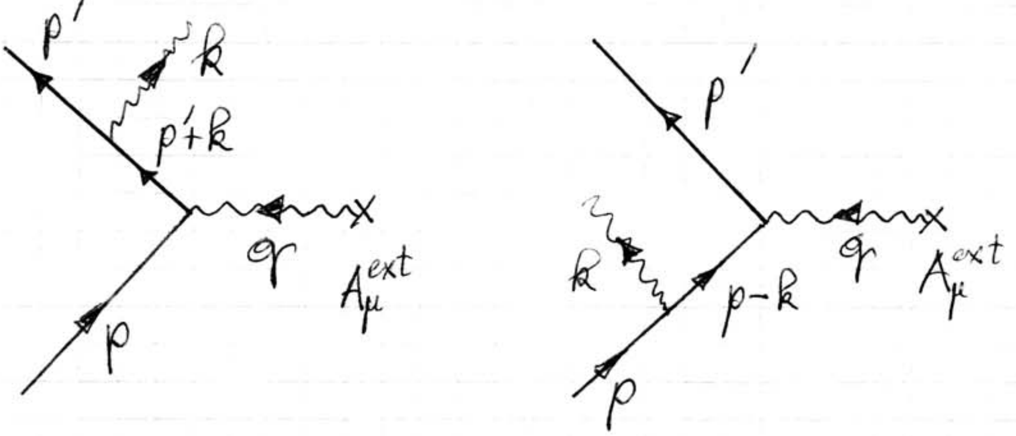
+ IR-finite contributions

$$\rightarrow \quad d\sigma(p \to p') = d\sigma(p \to p')\Big|_{tree} \times \Big[1 + \frac{\alpha}{\pi} \ln m_\gamma^2 \, f_{IR}(q^2) \Big]$$

$$+ \ldots$$

next step: <u>emission</u> of <u>real</u> photon

IR-divergences in $e^-(p)$ + external field $\rightarrow e^-(p') + \gamma(k)$



$$i\mathcal{M}(p \to p' + k) = \Big\{ \bar{u}(p')\, ie\gamma^\nu \frac{i}{\not{p}' + \not{k} - m + i\varepsilon}\, ie\gamma^\mu u(p)$$

$$+ \bar{u}(p')\, ie\gamma^\mu \frac{i}{\not{p} - \not{k} - m + i\varepsilon}\, ie\gamma^\nu u(p) \Big\} \tilde{A}^{ext}_\mu(\vec{q})\ \varepsilon_\nu(k)^*$$

$$= -ie^2 \bar{u}(p') \Big\{ \gamma^\nu \frac{\not{p}' + \not{k} + m}{2p' \cdot k} \gamma^\mu - \gamma^\mu \frac{\not{p} - \not{k} + m}{2p \cdot k} \gamma^\nu \Big\} u(p)$$

$$\cdot\, \tilde{A}^{ext}_\mu(\vec{q})\ \varepsilon_\nu(k)^*$$

(remark: factor $2\pi\,\delta(q^0)$ was split off according to definition of $\mathcal{M}(p \to p' + k)$)

Dirac equation can be used to simplify this expression:

$$\to\ i\mathcal{M}(p \to p' + k) = ie^2 \bar{u}(p') \Big\{ \gamma^\mu \Big(\frac{p^\nu}{p \cdot k} - \frac{p'^\nu}{p' \cdot k} \Big)$$

$$- \frac{\gamma^\mu \not{k} \gamma^\nu}{2p \cdot k} - \frac{\gamma^\nu \not{k} \gamma^\mu}{2p' \cdot k} \Big\} u(p)\ \tilde{A}^{ext}_\mu(\vec{q})\ \varepsilon_\nu(k)^*$$

check gauge-invariance: $\varepsilon_\nu(k) \to k_\nu$

now apply formula from p. 23/5 for $n=2$:

$$\sigma(e^- + \text{external field} \to e^- + \gamma) =$$

$$= \frac{\pi}{|\vec{p}|} \int d\mu(p')\, d\mu(k)\, \delta(p'^0 + k^0 - p^0)\, |\mathcal{M}(p \to p' + k)|^2$$

$$= \frac{1}{(4\pi)^2} \int d\Omega\, d\mu(k)\, \frac{|\vec{p}\,'|}{|\vec{p}|}\, \Theta(p^0 - k^0 - m)\, |\mathcal{M}(p \to p' + k)|^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2} \int d\mu(k)\, \frac{|\vec{p}\,'|}{|\vec{p}|}\, \Theta(p^0 + k^0 - m)\, |\mathcal{M}(p \to p' + k)|^2$$

observation: $\mathcal{M}(p \to p' + k) = \frac{a}{E_\gamma} + \mathcal{O}(1) \quad (E_\gamma = k^0)$

$$\frac{|\vec{p}\,'|}{|\vec{p}|} = 1 + \mathcal{O}(E_\gamma)$$

<u>if</u> only interested in IR-divergent part $\to$

$\to$ keep only terms $\sim \frac{1}{E_\gamma}$ in $\mathcal{M}(p \to p' + k)$:

$$\int d\mu(k)\, \frac{|\vec{p}\,'|}{|\vec{p}|}\, |\mathcal{M}(p \to p' + k)|^2 \sim \int dE_\gamma \left(\underbrace{\frac{1}{E_\gamma}}_{\text{logarithmic IR-divergence}} + \underbrace{\mathcal{O}(1)}_{\text{IR-finite}} \right)$$

$$i\mathcal{M}(p\to p'+k) = \overbrace{\bar{u}(p')\, ie\gamma^\mu u(p)\, \tilde{A}^{ext}_\mu(\vec{q})}^{i\mathcal{M}(p\to p')|_{tree}}$$

$$\times\, e\, \varepsilon_\nu(k)^* \left(\frac{p^\nu}{p\cdot k} - \frac{p'^\nu}{p'\cdot k}\right) + \dots$$

$$\Rightarrow \sum_\lambda |\mathcal{M}(p\to p'+k)|^2 = \left|\mathcal{M}(p\to p')\big|_{tree}\right|^2$$

$$\times\, e^2 \left[\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}\right] + \dots$$

$$\Rightarrow d\sigma(p\to p'+k) = d\sigma(p\to p')\big|_{tree} \times$$

$$\times \int d\mu(k)\, \Theta(p^0 - k^0 - m)\, e^2 \left[\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}\right]$$

$$+ \dots$$

$$= d\sigma(p\to p')\big|_{tree} \times \frac{e^2}{(2\pi)^3 2} \int_{\underbrace{m_\gamma}_{\text{IR-regulator}}}^{E_\ell} \frac{dE_\gamma}{E_\gamma}\, d\Omega_{\vec{k}} \quad \times$$

$$\times \left[\frac{2p\cdot p'}{(p\cdot\hat{k})(p'\cdot\hat{k})} - \frac{m^2}{(p\cdot\hat{k})^2} - \frac{m^2}{(p'\cdot\hat{k})^2}\right] \quad \text{where } k = E_\gamma \cdot \hat{k}$$

$$\int d\Omega_{\vec{k}} \, \frac{1}{(p.\hat{k})^2} = \frac{4\pi}{m^2} \quad \text{(exercise)}$$

Feynman parametrization for first term:

$$\int d\Omega_{\vec{k}} \, \frac{1}{(p.\hat{k})\,(p'.\hat{k})} =$$

$$= \int_0^1 d\xi \int d\Omega_{\vec{k}} \, \frac{1}{[\xi p.\hat{k} + (1-\xi) p'.\hat{k}]^2}$$

$$= 4\pi \int_0^1 d\xi \, \frac{1}{[\xi p + (1-\xi) p']^2}$$

$$= 4\pi \int_0^1 d\xi \, \frac{1}{m^2 - \xi(1-\xi) q^2}$$

$$\Rightarrow \quad d\sigma(p \to p' + k) = -\frac{\alpha}{\pi} \, d\sigma(p \to p')\big|_{tree} \, f_{IR}(q^2)$$

$$\Rightarrow \quad d\sigma(p \to p')\big|_{observable} = d\sigma(p \to p') + d\sigma(p \to p' + k)$$

indeed IR-finite