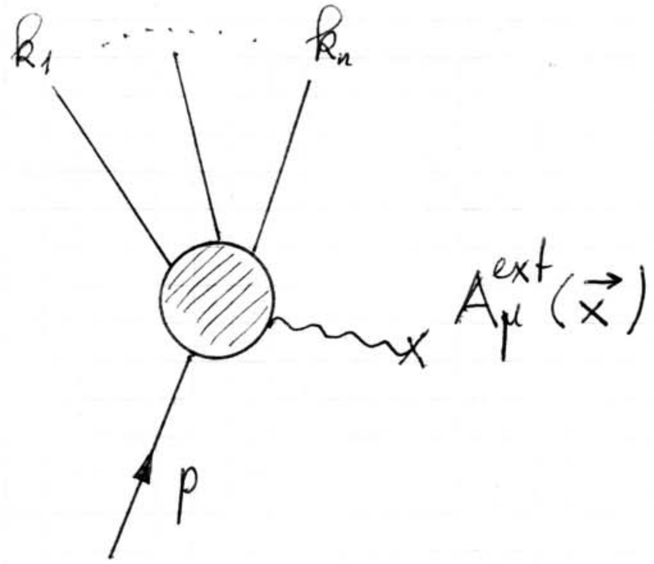
23. Cross-section formula for the scattering by a time-independent potential



potential time-independent $\Rightarrow$ energy conservation $\Rightarrow$ S-matrix element can be written in the form

$$\langle k_1, \dots, k_n \text{ out} \,|\, p \text{ in} \rangle = 2\pi i \,\delta(k_1^0 + \dots + k_n^0 - p^0)\, \mathcal{M}$$

(spin indices suppressed)

further approach in complete analogy to section 10:

$$\langle k_1, \dots, k_n \text{ out} \,|\, \psi \text{ in} \rangle_{A_\mu^{ext}} =$$

$$= \int d\mu(p)\, \langle k_1, \dots, k_n \text{ out} \,|\, p \text{ in} \rangle\, \psi(p)$$

momentum-space wave function concentrated around $\vec{p} = \vec{p}_0$

region of n-particle "phase-space"

$$w\left(|\psi\rangle \to B\right) =$$

$$= \int_B d\mu(k_1) \ldots d\mu(k_n) \langle \psi \text{ in} | k_1, \ldots, k_n \text{ out} \rangle_{A_\mu^{ext}}$$

$$\times \langle k_1, \ldots, k_n \text{ out} | \psi \text{ in} \rangle_{A_\mu^{ext}}$$

$$= \langle \psi \text{ in} | P_B | \psi \text{ in} \rangle$$

(for indistinguishable particles in final state $\to$ additional factor $\frac{1}{n!}$)

$$P_B = \int_B d\mu(k_1) \ldots d\mu(k_n) \, |k_1, \ldots, k_n \text{ out} \rangle \langle k_1, \ldots, k_n \text{ out}|$$

projection operator

$$N_{sc}\left(|\psi\rangle \to B\right) = \frac{N}{F} \underbrace{\int_F d^2\xi \, w\left(|\psi_{\vec{\xi}}\rangle \to B\right)}_{\sigma\left(|\psi\rangle \to B\right)}$$

$$\psi_{\vec{\xi}}(p) = e^{-i\vec{p}\cdot\vec{\xi}}\,\psi(p)$$

$$\sigma(|\psi\rangle \to B) = \int d^2_{\vec{\xi}}\; w\,(|\psi_{\vec{\xi}}\rangle \to B)$$

$F \to \mathbb{R}^2$ (potential concentrated in finite volume)

$$= \int d^2_{\xi} \langle \psi_{\vec{\xi}}\ in | P_B | \psi_{\vec{\xi}}\ in \rangle$$

$$= \int d^2_{\xi}\, d\mu(p')\, d\mu(p) \langle \psi_{\vec{\xi}}\, in | p'\, in \rangle$$

$$\times \langle p'\, in | P_B | p\ in \rangle \langle p\ in | \psi_{\vec{\xi}}\ in \rangle$$

$$= \int d^2_{\xi}\, d\mu(p')\, d\mu(p)\; e^{i\vec{p}'\cdot\vec{\xi}}\, \psi(p')^* \langle p'\, in | P_B | p\, in \rangle$$

$$e^{-i\vec{p}\cdot\vec{\xi}}\, \psi(p)$$

$\perp \vec{p_0}$

$$= \int d\mu(p')\, d\mu(p)\; (2\pi)^2\, \delta^{(2)}(\vec{p_\perp}' - \vec{p_\perp})\; \psi(p')^*\, \psi(p)$$

$$\times \langle p'\, in | P_B | p\ in \rangle$$

$$= 2\pi \int d^4p'\, \delta(p'^2 - m^2)\, \Theta(p'^0)\, d\mu(p)\, d\mu(k_1) \ldots d\mu(k_n)$$

$$\delta^{(2)}(\vec{p_\perp}' - \vec{p_\perp})\, \delta(p'^0 - p^0)\;\; \psi(p')^*\, \psi(p)$$

$$\mathcal{M}(p' \to k_1 \ldots k_n)^*\;\; \mathcal{M}(p \to k_1, \ldots, k_n)\; \delta(k_1^0 + \ldots + k_n^0 - p^0)$$

$$(*) = \delta(p'^2 - m^2)\, \delta^{(2)}(\vec{p}_\perp{}' - \vec{p}_\perp)\, \delta(p'^0 - p^0) =$$

$$= \delta[\underbrace{(p^0)^2 - \vec{p}_\perp{}^2}_{(p_\parallel)^2} - (p_\parallel')^2 - m^2)\, \delta^{(2)}(\vec{p}_\perp{}' - \vec{p}_\perp)$$

$$\delta(p'^0 - p^0)$$

$$= \frac{1}{2p_\parallel}\,[\delta(p_\parallel' - p_\parallel) + \delta(p_\parallel' + p_\parallel)]\,\delta^{(2)}(\vec{p}_\perp{}' - \vec{p}_\perp)\,\delta(p'^0 - p^0)$$

overlap of wave function only for

$$p_\parallel' = p_\parallel \Rightarrow (*) = \frac{1}{2p_\parallel}\,\delta^{(4)}(p' - p)$$

$$\Rightarrow \sigma(|\psi\rangle \to B) = \int d\mu(p)\, \underbrace{\Theta(p^0)}_{> 0\,\checkmark}$$

$$\times\, d\mu(k_1) \dots d\mu(k_n)\, \frac{\pi}{p_\parallel}\, \psi(p)^* \psi(p)$$

$$|\mathcal{M}(p \to k_1, \dots, k_n)|^2\, \delta(k_1^0 + \dots + k_n^0 - p^0)$$

wave function concentrated around $\vec{p} = \vec{p}_0$

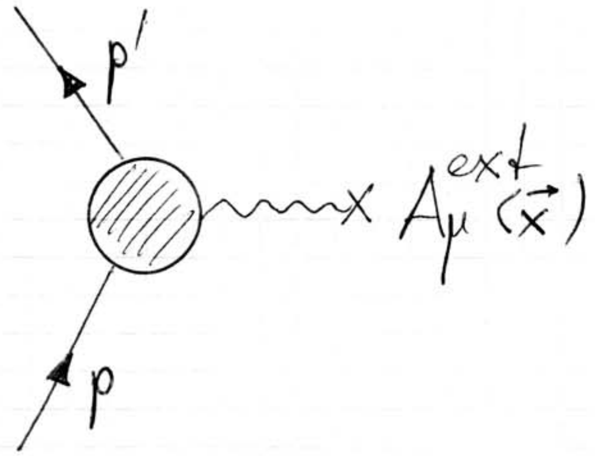
$$\Rightarrow \sigma(|\psi\rangle \to B) = \frac{\pi}{|\vec{p}_0|} \int_B d\mu(k_1) \dots d\mu(k_n)\, \delta(k_1^0 + \dots + k_n^0 - p^0)$$

$$\times\, |\mathcal{M}(p_0 \to k_1, \dots, k_n)|^2\, \underbrace{\int d\mu(p)\, |\psi(p)|^2}_{1}$$

$$\Rightarrow \sigma(p \to B) = \frac{\pi}{|\vec{p}|} \int_B d\mu(k_1) \dots d\mu(k_n)\, \delta(k_1^0 + \dots + k_n^0 - p^0) \times |\mathcal{M}(p \to k_1, \dots, k_n)|^2$$

(remark: add statistical factor $1/n!$ in case of indistinguishable particles)

special case $n = 1$:



$$p^2 = p'^2 = m^2$$

$$\sigma(p \to p') = \frac{\pi}{|\vec{p}|} \int \frac{d^3p'}{(2\pi)^3 2p'^0}\, \delta(p'^0 - p^0)\, |\mathcal{M}(p \to p')|^2$$

$$= \frac{1}{16\pi^2} \int d\Omega\, |\mathcal{M}(p \to p')|^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}(p \to p')|^2}{(4\pi)^2} \quad \text{differential cross-section}$$

back to electron scattering by an external electromagnetic field:

$$\langle p', s' \text{ out} | p, s \text{ in} \rangle_{A_\mu^{ext}} =$$

$$= ie\, \bar{u}(p', s') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p,s)$$

$$\times \quad \frac{\tilde{A}_\mu^{ext}(q)}{1 - \hat{\Pi}(q^2)} \quad , \quad q = p' - p$$

A_μ time-independent $\Rightarrow$ $\tilde{A}_\mu^{ext}(q) =$

$$= 2\pi\, \delta(q^0) \underbrace{\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} A_\mu^{ext}(\vec{x})}_{=: \tilde{A}_\mu(\vec{q})}$$

$$\Rightarrow \mathcal{M}(p \to p') = ie\, \bar{u}(p', s') \left[\gamma^\mu F_1(-\vec{q}^{\,2}) + \right.$$

$$\left. + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(-\vec{q}^{\,2}) \right] u(p,s) \quad \frac{\tilde{A}_\mu(\vec{q})}{1 - \hat{\Pi}(-\vec{q}^{\,2})}$$

$(q^2 = -\vec{q}^{\,2}$ as $q^0 = 0)$

$$\tilde{A}_\mu^{eff}(\vec{q}) := \frac{\tilde{A}_\mu(\vec{q})}{1 - \hat{\Pi}(-\vec{q}^2)}$$ can be interpreted

as an "effective" electromagnetic potential (includes the effect of vacuum polarization)

transforming back to $\vec{x}$-space:

$$A_\mu^{eff}(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \, e^{+i\vec{q}\cdot\vec{x}} \, \tilde{A}_\mu^{eff}(\vec{q})$$

Coulomb potential $\quad \phi(\vec{x}) = \dfrac{Q}{4\pi |\vec{x}|}$

$\longrightarrow$ Uehling potential

$$\phi^{eff}(\vec{x}) = \frac{Q}{4\pi r} \times \begin{cases} 1 + \dfrac{\alpha}{3\pi}\left[\ln\dfrac{1}{(mr)^2} - 2\gamma - \dfrac{5}{3} + \dots\right] & mr \ll 1 \\ 1 + \dfrac{\alpha}{4\sqrt{\pi}\,(mr)^{3/2}} \, e^{-2mr} & mr \gg 1 \end{cases}$$