

## 22. Electron scattering by an external field

$$A_\mu(x) \rightarrow A_\mu(x) + A_\mu^{\text{ext}}(x)$$

↑                            ↗

quantum field                 external (classical) elec.  
field

compute electron two-point function in the presence  
of  $A_\mu^{\text{ext}}$  (considering only terms linear in  $A_\mu^{\text{ext}}$ )

$$\rightarrow \langle p', s' \text{ out} | p, s \text{ in} \rangle_{A_\mu^{\text{ext}}} =$$

$$= \bar{u}(p', s') i e [g^\mu + \tilde{\lambda}^\mu(p', p)] u(p, s)$$

$$\times [\tilde{A}_\mu^{\text{ext}}(q) + \frac{-i}{q^2} i \underbrace{[\hat{\Pi}_\mu^\nu(q) A_\nu^{\text{ext}}(q)]}_{(q^2 g_\mu^\nu - q_\mu q^\nu) \hat{\Pi}(q^2)}$$

$$q = p' - p$$

$$\tilde{A}_\mu^{\text{ext}}(q) = \int d^4x e^{iqx} A_\mu^{\text{ext}}(x) \quad \text{Fourier transform of ext. field}$$

$$\left\langle p', s' \text{out} | p, s \text{ in} \right\rangle \Big|_{A_\mu^{\text{ext}}} =$$

$$= \bar{u}(p, s) i \epsilon [g^\mu + \tilde{\lambda}^\mu(p', p)] u(p, s) \frac{\tilde{A}_\mu^{\text{ext}}(q)}{1 - \hat{\Pi}(q^2)}$$

we have to compute

$$\tilde{\lambda}^\mu(p', p) = [\lambda^\mu(p', p) - \lambda^\mu(p, p)]_{p'^2 = p'^2 = m^2}$$

where

$$i \lambda^\mu(p', p) = e^2 \int \frac{d^d k}{(2\pi)^d} \frac{-g^\alpha (p' + k + m) g^\mu (p + k + m) g_\alpha}{(k^2 - m_g^2 + i\varepsilon) [(k+p)^2 - m^2 + i\varepsilon] [(k+p')^2 - m^2 + i\varepsilon]}$$

$$\frac{1}{abc} = 2! \int_0^1 dx dy dz \delta(x+y+z-1) \times \frac{1}{(xa+yb+zc)^3}$$

$$a = k^2 - m_g^2 + i\varepsilon, b = (k+p)^2 - m^2 + i\varepsilon, c = (k+p')^2 - m^2 + i\varepsilon$$

$$xa + yb + zc =$$

$$= R^2 + 2(yp + zp') \cdot R - x m_y^2 + y \underbrace{(p^2 - m^2)}_0 + z \underbrace{(p'^2 - m^2)}_0$$

$$= (R + yp + zp')^2 - (yp + zp')^2 - x m_y^2 + i\varepsilon$$

shift  $R \rightarrow R - yp - zp' \rightarrow$

$$iN(p/p) \Big|_{\substack{p^2 = p'^2 = m^2 \\ UV\text{-div.}}} = 2e^2 \int_0^1 dx dy dz \delta(x+y+z-1) \\ \times \frac{dR}{(2\pi)^d} \underbrace{\frac{R^2}{d} g^\alpha g^\beta g^\mu g_\beta^\nu g_\alpha^\lambda}_{[R^2 - (yp + zp')^2 - x m_y^2 + i\varepsilon]^3} \underbrace{g^\alpha [p'(1-z) - py + m] g^\mu [p(1-y) - p'z + m] g_\lambda}_\text{UV-finite}$$

$$g^\alpha g^\beta g^\mu g_\beta^\nu g_\alpha^\lambda = (2-d) g^\alpha g^\mu g_\alpha^\lambda = (2-d)^2 g^\mu$$

$$g^\alpha [p'(1-z) - py + m] g^\mu [p(1-y) - p'z + m] g_\lambda \\ = -2 [p(1-y) - p'z] g^\mu [p'(1-z) - py] \\ + 4m [p(1-2y) + p'(1-2z)]^\mu - 2m^2 g^\mu$$

$$\bar{u}(p', s') \Lambda^\mu(p', p) \Big|_{p^2 = p'^2 = m^2} u(p, s)$$

→ use  $p' u(p, s) = m u(p, s)$  and

$$\bar{u}(p', s') p' = m \bar{u}(p', s')$$

exercise: check gauge invariance

$$\bar{u}(p', s') \Lambda^\mu(p', p) \Big|_{p^2 = p'^2 = m^2} u(p, s) (p' - p)_\mu = 0$$

hint:  $R^2 - (y p + z p')^2 - x m_p^2 + i\epsilon$

$$= R^2 - (y^2 + z^2)m^2 - 2yz p \cdot p' - x m_p^2 + i\epsilon$$

Symmetric under  $y \leftrightarrow z$

exercise: check  $S_1 = S_2$  (equivalent to  $Z_1 = Z_2$ )

exercise: show (Gordon decomposition)

$$\bar{u}(p') g^\mu u(p) = \bar{u}(p') \left[ \frac{p^\mu + p'^\mu}{2m} + \frac{i \sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

$$\sigma^{\mu\nu} := \frac{i}{2} [g^\mu, g^\nu], \quad q = p' - p$$

after a lengthy, but straightforward calculation,  
one finds :

$$\bar{u}(p', s') \tilde{\Lambda}^\mu(p', p) \Big|_{\substack{p^2 = p'^2 = m^2}} u(p, s) =$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx dy dz \delta(x+y+z-1)$$

$$\begin{aligned} & \bar{u}(p', s') \left\{ g^\mu \left[ \ln \frac{(1-x)^2 m^2}{(1-x)^2 m^2 - q^2 y z} + \right. \right. \\ & + \frac{m^2 (1-4x+x^2) + q^2 (1-y)(1-z)}{(1-x)^2 m^2 - q^2 y z + x m_y^2} - \frac{m^2 (1-4x+x^2)}{(1-x)^2 m^2 + x m_y^2} \left. \right] \\ & \left. + \frac{i \sigma^{\mu\nu} q_\nu}{2m} \frac{2m^2 \times (1-x)}{m^2 (1-x)^2 - q^2 y z} \right\} u(p, s) \end{aligned}$$

$$\begin{aligned} \langle p', s' \text{ out} | p, s \text{ in} \rangle_{A_\mu^{\text{ext}}} &= i e \bar{u}(p', s') \underbrace{[g^\mu + \tilde{\Lambda}^\mu(p', p)]}_{g^\mu F_1(q^2) + \frac{i \sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)} u(p, s) \\ & \times \frac{\tilde{A}_\mu^{\text{ext}}(q)}{1 - \hat{\Pi}(q^2)} \end{aligned}$$

form factors  $F_{1,2}(q^2)$ ,  $F_1(0)=0$  (charge renorm.)

$F_2(0) \rightarrow$  anomalous magnetic moment of  
the electron

$$\begin{aligned} F_2(0) &= \frac{\alpha}{\pi} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{x}{1-x} \\ &= \frac{\alpha}{\pi} \int_0^1 dx x = \frac{\alpha}{2\pi} \quad \text{Schwinger 1948} \\ &\uparrow \quad \text{exercice} \quad \text{Phys. Rev. } 73 \text{ (1948)} 416L \end{aligned}$$

physical interpretation of the form factors (for  $q^2 \rightarrow 0$ ):

consider nonrelativistic limit of  $\langle p', s' \text{out} | p, s \text{in} \rangle_{A_\mu^{\text{ext}}}$

for a purely electric, time-independent field

$$A_\mu^{\text{ext}}(t, \vec{x}) = (\phi(\vec{x}), \vec{o})$$

$$\begin{aligned} \Rightarrow \tilde{A}_0^{\text{ext}}(q) &= \int d^4x e^{iq \cdot x} \phi(\vec{x}) = \\ &= 2\pi \delta(q^0) \underbrace{\int d^3x e^{-i\vec{q} \cdot \vec{x}} \phi(\vec{x})}_{=: \tilde{\phi}(\vec{q})} \end{aligned}$$

(22/4)

$$\langle p', s' \text{ out} | p, s \text{ in} \rangle \xrightarrow{\text{nonrelat.}} i e F_1(0) \pi(p', s') g^* \mu(p, s) \\ \times \tilde{\phi}(\vec{q}) 2\pi \delta\left(\frac{\vec{p}'^2}{2m} - \frac{\vec{p}^2}{2m}\right) \\ + O(q)$$

$$\pi(p', s') g^* \mu(p, s) = \mu(p, s')^\dagger \mu(p, s) + O(q) \\ \simeq 2m \xi(s')^\dagger \xi(s) \quad (\text{Particle Physics I})$$

nonrelativistic scattering theory (potential energy  $V$ ):

$$\langle \vec{p}', s' \text{ out} | \vec{p}, s \text{ in} \rangle = (2\pi)^3 2m \delta^{(3)}(\vec{p}' - \vec{p}) \xi(s')^\dagger \xi(s) \\ - 2\pi i \delta\left(\frac{\vec{p}'^2}{2m} - \frac{\vec{p}^2}{2m}\right) \langle \vec{p}', s' | V | \vec{p}, s \rangle$$

↑      ↑  
momentum eigenstates  
of free theory  
(relativistic normalization!)

(Born approximation)

$$V \text{ spin-independent} \rightarrow \langle \vec{p}', s' | V | \vec{p}, s \rangle = \\ = 2m \xi(s')^\dagger \xi(s) \int d^3x e^{-i\vec{p}'\vec{x}} V(\vec{x}) e^{i\vec{p}\vec{x}}$$

$$= 2m \xi(s')^\dagger \xi(s) \tilde{V}(\vec{q})$$

comparison with our result  $\rightarrow V(\vec{x}) = -e \underbrace{F_1(0)}_1 \phi(\vec{x})$   
 $e^-!$

analogous procedure for pure (time-indep.)  
magnetic field:

$$A_\mu^{\text{ext}}(t, \vec{x}) = (0, \vec{A}(\vec{x}))$$

↑  
 time-independent vector potential

$$\vec{B}(\vec{x}) = \text{rot } \vec{A}(\vec{x})$$

$$\langle p', s' \text{out} | p, s \text{in} \rangle_{\vec{B}(\vec{x})} = \quad \text{Gordon decomposition}$$

$$= ie \bar{u}(p', s') \left\{ \frac{p^\mu + p'^\mu}{2m} F_1(q^2) + \frac{i}{2m} [F_1(q^2) + F_2(q^2)] \sigma^{\mu\nu} q_\nu \right\} u(p, s)$$

$$* \tilde{A}_\mu^{\text{ext}}(q)$$

↑  
 does not "shake" the spin of  
 the electron (corresponds to effect  
 of Lorentz force for spinless particle)

we consider the expression

$$\text{ie } \frac{i}{2m} [F_1(q^2) + F_2(q^2)] \tilde{A}_\mu^{\text{ext}}(q)$$

$$\bar{u}(p', s') \sigma^{\mu\nu} q_\nu u(p, s)$$

in the nonrelativistic limit

$$\tilde{A}_\mu^{\text{ext}}(q) = \int d^4x A_\mu^{\text{ext}}(x) e^{+iq \cdot x}$$

$$= \int d^4x (0, \vec{A}(\vec{x})) e^{i\vec{q} \cdot \vec{x}} e^{-i\vec{q} \cdot \vec{x}}$$

$$= 2\pi \delta(\vec{q}) (0, \int d^3x \vec{A}(\vec{x}) e^{-i\vec{q} \cdot \vec{x}})$$

$$\bar{u}(p', s') \sigma^{\mu\nu} q_\nu u(p, s) \xrightarrow{\text{nonrelat.}}$$

$$\rightarrow m [\xi(s')^\dagger, \xi(s')^\dagger] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma^{ij} q_j \begin{bmatrix} \xi(s) \\ \xi(s) \end{bmatrix}$$

$$\varepsilon^{ijk} \begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix}$$

$$(\varepsilon^{123} = +1)$$

$$\Rightarrow \tilde{A}_\mu(q) \bar{u}(p, s') \sigma^\mu q, u(p, s)$$

nonrelat.

$$\rightarrow 2\pi S(q^0) \int d^3x A^i(\vec{x}) e^{-i\vec{q} \cdot \vec{x}} q^j$$

$$\times 2m \xi(s') + \sigma^k \xi(s) \varepsilon^{ijk}$$

$$= 2\pi S(q^0) \int d^3x A^i(\vec{x}) i \frac{\partial}{\partial x^j} e^{-i\vec{q} \cdot \vec{x}}$$

$$2m \xi(s') + \sigma^k \xi(s) \varepsilon^{ijk}$$

partial int.

$$\stackrel{\downarrow}{=} 2\pi i S(q^0) \int d^3x \underbrace{\varepsilon^{kji} \frac{\partial A^i(\vec{x})}{\partial x^j}}_{(\vec{\nabla} \times \vec{A})^k = \vec{B}^k} e^{-i\vec{q} \cdot \vec{x}}$$

$$\times 2m \xi(s') + \sigma^k \xi(s)$$

$$= 2\pi i S(q^0) \int d^3x \vec{B}(\vec{x}) e^{-i\vec{q} \cdot \vec{x}} 2m \xi(s') + \vec{\sigma} \xi(s)$$

$$\Rightarrow \langle p/s' \text{ out} | p, s \text{ in} \rangle_{\vec{B}(\vec{x})} \xrightarrow{\text{nonrelat.}}$$

$$\text{i.e. } \frac{i}{2m} [\bar{F}_1(0) + \bar{F}_2(0)]$$

$$2\pi i S(q^0) \int d^3x \vec{B}(\vec{x}) e^{-i\vec{q} \cdot \vec{x}} 2m \xi(s') + \vec{\sigma} \xi(s)$$

→ scattering in a potential

$$V(\vec{x}) = -\vec{\mu} \cdot \vec{B}(\vec{x})$$

magnetic moment

$$\vec{\mu} = -\frac{e}{2m} \underbrace{[F_1(0) + F_2(0)]}_{1} \vec{\sigma}$$

$$= -\frac{e}{2m} \underbrace{2}_{e^{-1}} \underbrace{[1+F_2(0)]}_{\mu_B} \underbrace{\frac{\vec{\sigma}}{2}}_{g} \underbrace{\vec{s}}_{\text{Bohr magneton} \quad \text{Landé factor} \quad \text{spin}}$$

true approximation :  $g = 2$

higher order corrections →  $g = 2 [1+F_2(0)]$

$$\begin{aligned} \text{anomalous magnetic moment } \alpha_e &:= \frac{g-2}{2} = F_2(0) \\ &= \frac{\alpha}{2\pi} + O(\alpha^2) \end{aligned}$$

including also coupling to  $\vec{L}$ :

$$\vec{\mu} = -\frac{e}{2m} \left\{ \vec{L} + 2 [1 + F_2(0)] \frac{\vec{\sigma}}{2} \right\}$$

higher order corrections:

$$F_2(0)^{\text{QED}} = \alpha_e^{\text{QED}} = \frac{\alpha}{2\pi} - 0.328\ 478\ 444\ 002\ 90(60) \left(\frac{\alpha}{\pi}\right)^2$$

$$+ 1.181\ 234\ 016\ 828(19) \left(\frac{\alpha}{\pi}\right)^3$$

$$- 1.9144(35) \left(\frac{\alpha}{\pi}\right)^4$$

$$+ 0.0(3.8) \left(\frac{\alpha}{\pi}\right)^5$$

from: F. Jegerlehner, The Anomalous Magnetic Moment of the Muon, Springer, 2008

$$\alpha_e^{\text{SM}} = \alpha_e^{\text{QED}} + 1.706(30) \times 10^{-12}$$

hadronic + weak contributions

$$\alpha_e^{\text{exp}} = 0.001\ 159\ 652\ 180\ 85(76)$$

$$\Rightarrow \alpha^{-1}(\alpha_e) = 137.035\ 999\ 069(90)(12)(30)(3)$$