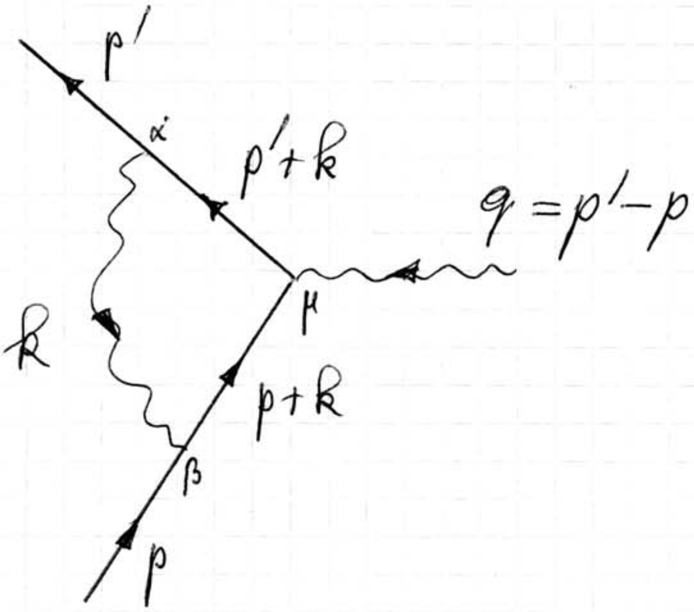
20. Vertex correction in QED



$$i e \Lambda^\mu(p',p) = \int \frac{d^d k}{(2\pi)^d}\, i e \gamma^\alpha \frac{i}{\not p' + \not k - m + i\varepsilon}\, i e \gamma^\mu$$

$$\times \frac{i}{\not p + \not k - m + i\varepsilon} \times i e \gamma^\beta \; \frac{-i g_{\alpha\beta}}{k^2 - m_\gamma^2 + i\varepsilon}$$

$$= e^3 \int \frac{d^d k}{(2\pi)^d}\, \gamma^\alpha (\not p' + \not k + m)\, \gamma^\mu\, (\not p + \not k + m)\, \gamma_\alpha$$

$$\times \frac{1}{k^2 - m_\gamma^2 + i\varepsilon}\; \frac{1}{(p+k)^2 - m^2 + i\varepsilon}\; \frac{1}{(p'+k)^2 - m^2 + i\varepsilon}$$

$$\Lambda^\mu(p',p) = \underbrace{\Lambda^\mu(p',p) - \Lambda^\mu(p,p)}_{\text{\underline{UV-finite}}} + \Lambda^\mu(p,p)$$

$$\frac{\not p + \not q + \not k + m}{(k+p+q)^2 - m^2} - \frac{\not p + \not k + m}{(k+p)^2 - m^2} =$$

$$= \frac{(\not p + \not q + \not k + m)\left[(k+p)^2 - m^2\right] - (\not p + \not k + m)\left[(k+p+q)^2 - m^2\right]}{\left[(k+p+q)^2 - m^2\right]\left[(k+p)^2 - m^2\right]}$$

$$= \Big\{ \cancel{(\not p + \not k + m)\left[(k+p)^2 - m^2\right]} + \not q \left[(k+p)^2 - m^2\right]$$

$$- \cancel{(\not p + \not k + m)\left[(k+p)^2 - m^2\right]} - (\not p + \not k + m)\, 2\,(k+p).q \Big\}$$

$$\times \frac{1}{\left[(k+p+q)^2 - m^2\right]\left[(k+p)^2 - m^2\right]} \sim \frac{1}{\not k^2}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{\not k}{k^2\, k^2\, k^2} \quad \text{finite}$$

explicit calculation gives

$$\Lambda^\mu(p,p) = \underbrace{\frac{e^2}{(4\pi)^2\varepsilon}}_{p\text{-independent}} \gamma^\mu + \text{finite contributions}$$

→ we perform the following <u>subtraction</u>:

$$\Lambda^\mu(p',p) = \underbrace{\Lambda^\mu(p',p) - \Lambda^\mu(p,p)\Big|_{p^2=m^2}}_{UV\text{-finite}} + \underbrace{\Lambda^\mu(p,p)\Big|_{p^2=m^2}}_{UV\text{-div.}}$$

$$i\Lambda^\mu(p,p)\Big|_{p^2=m^2} =$$

$$= e^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\alpha \frac{1}{\not p + \not k - m + i\varepsilon} \gamma^\mu \frac{1}{\not p + \not k - m - i\varepsilon} \gamma_\alpha \frac{1}{k^2 - m_\gamma^2 + i\varepsilon}\Bigg|_{p^2=m^2}$$

compare with

$$-i\Sigma(p) = -e^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\alpha \frac{1}{\not p + \not k - m + i\varepsilon} \gamma_\alpha \frac{1}{k^2 - m_\gamma^2 + i\varepsilon}$$

$$\frac{\partial}{\partial p_\mu} \frac{1}{\not{p}+\not{k}-m} = -\frac{1}{\not{p}+\not{k}-m}\gamma^\mu \frac{1}{\not{p}+\not{k}-m}$$

$$\Rightarrow i\,\frac{\partial \Sigma(p)}{\partial p^\mu} = -e^2 \int \frac{d^dk}{(2\pi)^d}\, \gamma^\alpha \frac{1}{\not{p}+\not{k}-m+i\varepsilon}\, \gamma^\mu \frac{1}{\not{p}+\not{k}-m+i\varepsilon}\, \gamma_\alpha \frac{1}{k^2-m_\gamma^2+i\varepsilon}$$

$$= -i\,\Lambda^\mu(p,p)$$

$$\Rightarrow \Lambda^\mu(p,p)\Big|_{p^2=m^2} = \Lambda^\mu(p,p)\Big|_{p^2=m_{ph}^2} =$$

$$= -\frac{\partial \Sigma(p)}{\partial p_\mu}\Big|_{p^2=m_{ph}^2} = -(Z_2-1)\,\gamma^\mu$$

remember: $\frac{\partial \Sigma(p)}{\partial \not{p}}\Big|_{\not{p}=m_{ph}} = Z_2 - 1$

$$\gamma^\mu + \Lambda^\mu(p',p) =$$

$$= \gamma^\mu + \Lambda^\mu(p,p)\Big|_{p^2=m_{ph}^2} + \underbrace{\Lambda^\mu(p',p) - \Lambda^\mu(p,p)\Big|_{p^2=m_{ph}^2}}_{\tilde{\Lambda}(p',p)}$$

$$= [1-(Z_2-1)]\,\gamma^\mu + \tilde{\Lambda}^\mu(p',p)$$

$$= \frac{1}{1+(Z_2-1)}\,\gamma^\mu + \tilde{\Lambda}^\mu(p',p)$$

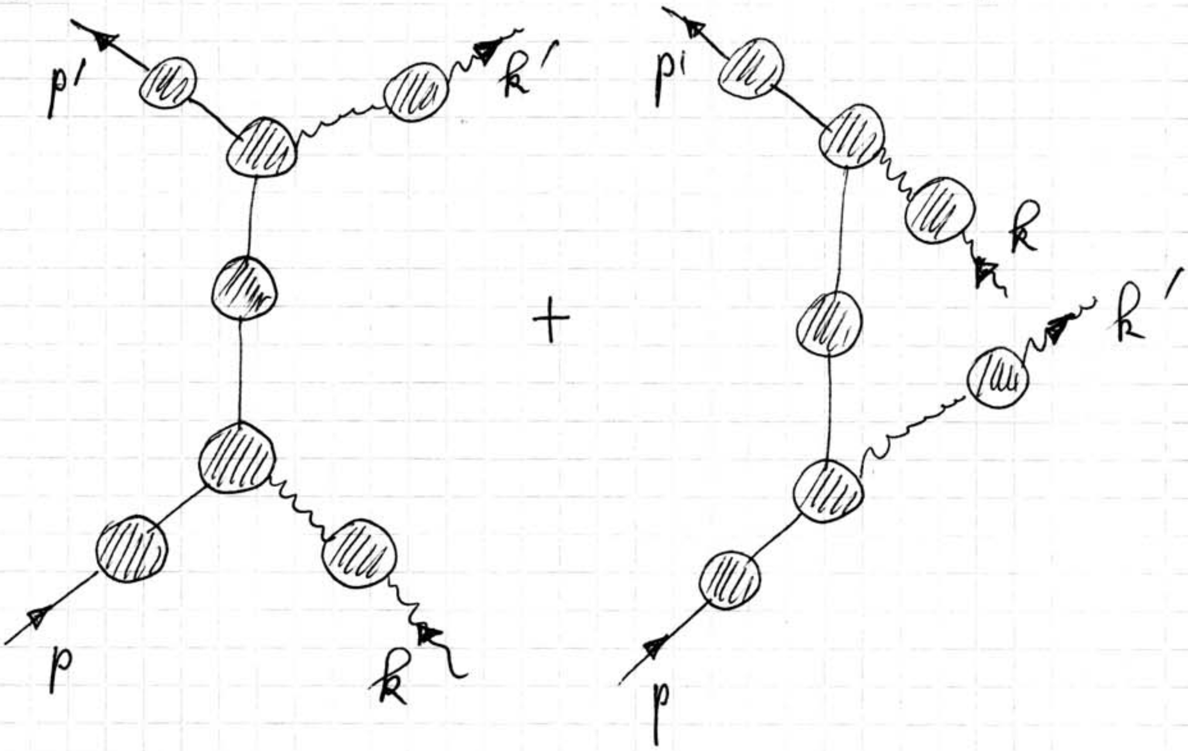
$$= \frac{1}{Z_2}\,\gamma^\mu + \tilde{\Lambda}^\mu(p',p)$$

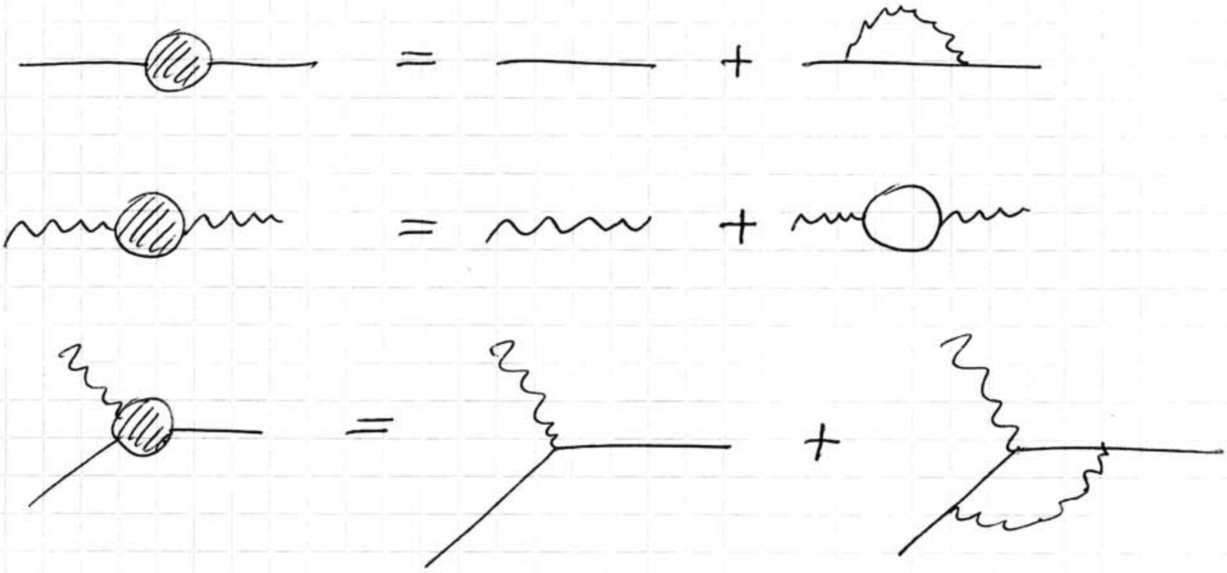
remark: sometimes, the vertex renormalization constant Z_1 is introduced,

$$\gamma^\mu + \Lambda^\mu(p',p) = \frac{1}{Z_1}\,\gamma^\mu + \tilde{\Lambda}^\mu(p,p')$$

and one finds that $Z_1 = Z_2$

charge renormalization $\quad e^- \gamma \to e^- \gamma$





$$i\mathcal{M}_{fi} = \sqrt{Z_2}\ \bar{u}(p')\ \Big\{ ie\left[\frac{1}{Z_1}\gamma^\nu + \tilde{\Lambda}^\nu(p', p+k)\right]$$

$$\frac{i\,Z_2}{\underbrace{\not p + \not k - m_{ph} + \dots}_{\text{finite contr.}}}\ ie\left[\frac{1}{Z_1}\gamma^\mu + \tilde{\Lambda}^\mu(p+k, p)\right]$$

$$+\ ie\left[\frac{1}{Z_1}\gamma^\mu + \tilde{\Lambda}^\mu(p', p-k')\right]$$

$$\frac{i\,Z_2}{\not p - \not k' - m_{ph} + \dots}\ ie\left[\frac{1}{Z_1}\gamma^\nu + \tilde{\Lambda}^\nu(p-k', p)\right]\Big\}$$

$$u(p)\ \sqrt{Z_2}\quad \sqrt{Z_3}\ \varepsilon_\mu(k)\ \sqrt{Z_3}\ \varepsilon_\nu(k')^*$$

$$\overset{Z_1 = Z_2}{=}\ -ie^2 Z_3\ \bar{u}(p')\Big\{ \left[\gamma^\nu + \tilde{\Lambda}^\nu(p', p+k)\right]$$

$$\frac{1}{\not p + \not k - m_{ph}}\left[\gamma^\mu + \tilde{\Lambda}^\mu(p+k, p)\right]$$

$$+\left[\gamma^\mu + \tilde{\Lambda}^\mu(p', p-k')\right]\frac{1}{\not p - \not k' - m_{ph} + \dots}$$

$$\left[\gamma^\nu + \tilde{\Lambda}^\nu(p-k', p)\right]\Big\}\ u(p)$$

$\mathcal{M}_{fi}$ UV-finite with charge renormalization

$$e_{phys}^2 = Z_3 \, e^2$$

remark: one-loop cross section $\sigma(i \to f)$ in general not IR-finite

IR-finite result obtained by adding $\sigma(i \to f \gamma_{soft})$ (only tree contribution) → emission of additional soft photon

$$\underbrace{\sigma(i \to f[\gamma])}_{\text{IR-finite}} = \underbrace{\sigma(i \to f)\Big|_{\text{one-loop}}}_{\text{IR-divergent}} + \underbrace{\sigma(i \to f \gamma_{soft})\Big|_{\text{tree}}}_{\text{IR-divergent}}$$

limit $k \to 0$ in Compton scattering

→ Thomson cross section

$$\sigma_{\text{Thomson}} = \frac{8\pi\alpha^2}{3 m_{ph}^2} \quad , \quad \alpha = \frac{e_{ph}^2}{4\pi}$$

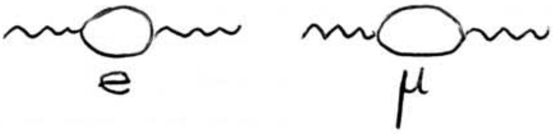
→ charge renormalization $e_{phys} = \sqrt{Z_3}\, e$ can be obtained by <u>defining</u> e_{phys} via the Thomson limit as a <u>reference process</u>

<u>why</u> $Z_1 = Z_2$? (follows from gauge invariance)

suppose $Z_1 \neq Z_2$: $e_{phys} = \frac{Z_2}{Z_1} \sqrt{Z_3}\, e$

QED with <u>several</u> species of charged fermions (e.g. $e^-, \mu^-, \dots$)

vacuum polarization receives contributions from all particles



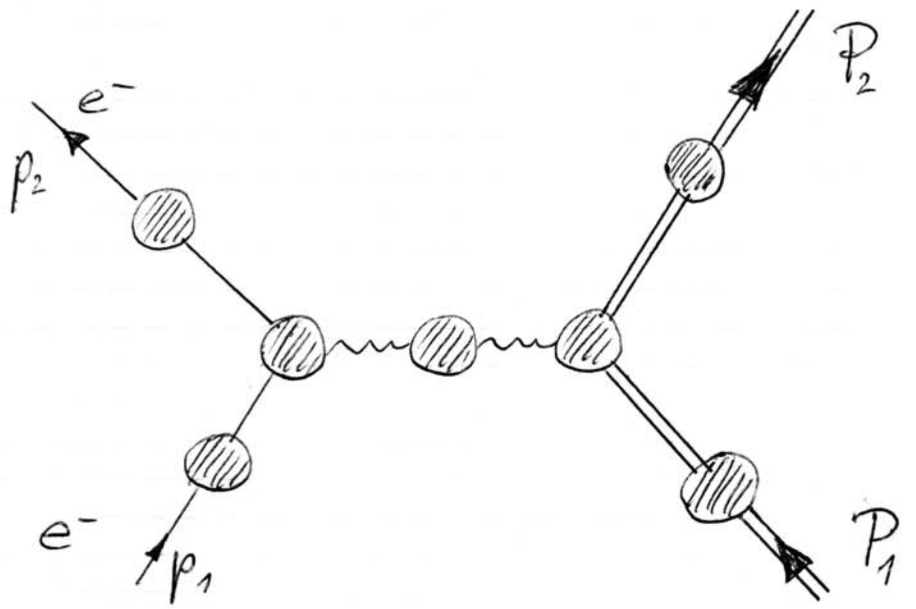
→ $Z_1^{(f)}, Z_2^{(f)}$ depend on m_f

$\frac{Z_2^{(f)}}{Z_1^{(f)}} \neq 1$ → ratio mass-dependent

→ $e_{phys}^{(f)} = \frac{Z_2^{(f)}}{Z_1^{(f)}} \sqrt{Z_3}\, e$ would depend on f

but $Z_2^{(f)} = Z_1^{(f)}\ \forall f$ guarantees <u>universal</u> $e_{phys} = \sqrt{Z_3}\, e$

further example : renormalization of $e^- f \to e^- f$ $(f \neq e)$ at one-loop



$$i\mathcal{M}_{fi} = \sqrt{Z_2}\,\bar{u}(p_2)\, ie \left[\frac{1}{Z_1}\gamma^\mu + \tilde{\Lambda}^\mu(p_2,p_1)\right] u(p_1)\sqrt{Z_2}$$

$$\times \frac{-i g_{\mu\nu} Z_3}{k^2(1-\hat{\Pi}(k^2))}$$

$$\times \sqrt{Z_2'}\,\bar{U}(P_2)\, ie\left[\frac{1}{Z_1'}\gamma^\nu + \tilde{\Lambda}'^\nu(P_2,P_1)\right] U(P_1)\sqrt{Z_2'}$$

$$\overset{Z_1=Z_2,\ Z_1'=Z_2'}{=} i\underbrace{e^2 Z_3}_{e^2_{phys}}\ \bar{u}(p_2)\left[\gamma^\mu + \tilde{\Lambda}^\mu(p_2,p_1)\right] u(p_1)$$

$$\frac{1}{k^2(1-\hat{\Pi}(k^2))}\ \bar{U}(P_2)\left[\gamma_\mu + \tilde{\Lambda}'(P_2,P_1)\right] U(P_1)$$

$$k = p_2 - p_1$$

<u>remark:</u> $\alpha_{eff}(k^2) := \dfrac{\alpha}{1 - \hat{\Pi}(k^2)}$

can be interpreted as a <u>k^2-dependent</u> effective coupling strength of the electro-magnetic interaction

$$\hat{\Pi}(k^2) = \frac{2\alpha}{\pi} \int_0^1 dx \; x(1-x) \ln \frac{m^2 - k^2 x(1-x) - i\varepsilon}{m^2}$$

(only e^--contribution considered here)

$$\hat{\Pi}(0) = 0 \Rightarrow \alpha_{eff}(0) = \alpha \simeq \frac{1}{137}$$

$$\hat{\Pi}(k^2) \xrightarrow[-k^2 \gg m^2]{} \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \left[\ln\left(-\frac{k^2}{m^2}\right) + \ln\left(x(1-x)\right) + \mathcal{O}\left(\frac{m^2}{k^2}\right) \right]$$

$$= \frac{\alpha}{3\pi} \left[\ln\left(-\frac{k^2}{e^{5/3} m^2}\right) + \mathcal{O}\left(\frac{m^2}{k^2}\right) \right]$$

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$$\Rightarrow \alpha_{eff}(k^2) \xrightarrow[-k^2 \gg m^2]{} \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln\left(-\frac{k^2}{e^{5/3} m^2}\right)}$$

"break-down" of QED at $\sqrt{|k^2|} \simeq \Lambda_{QED}$
(Landau pole)

$$\frac{\alpha}{3\pi} \ln \frac{\Lambda_{QED}^2}{e^{5/3} m_e^2} \simeq 1$$

$$\Rightarrow \quad \Lambda_{QED} \simeq m_e \, e^{\frac{3\pi}{2\alpha}} \approx 10^{277} \, GeV$$

energy, of course, beyond good and evil (gravitation already relevant at Planck energy $M_{Planck} \simeq 10^{19}$ GeV) $\rightarrow$ Λ_{QED} purely academic

range of applicability of QED ends already at considerably lower energies for physical reasons

example: $e^+ e^- \rightarrow \mu^+ \mu^-$