18. QED two-point function for fermions

we want to compute $\langle \Omega | T\, \psi(x)\, \psi(y) | \Omega \rangle$ in one-loop approximation (remember: $\psi(x)$ is a Heisenberg-operator)

for a change, we want to employ the path integral formalism for the perturbative expansion of Green functions:

$$S[\varphi] = \underset{\substack{\uparrow \\ \text{bilinear in } \varphi}}{S_0[\varphi]} + S_{int}[\varphi]$$

φ denotes the fields present in the theory under investigation (e.g. $\varphi = (\psi, \bar{\psi}, A^\mu, \dots)$)

$$\langle \Omega | T\, \phi(x_1) \dots \phi(x_n) | \Omega \rangle =$$

$$= \frac{\int [d\varphi]\, e^{iS[\varphi]}\, \varphi(x_1) \dots \varphi(x_n)}{\int [d\varphi]\, e^{iS[\varphi]}}$$

$$= \frac{\int [d\varphi]\, e^{iS_0[\varphi]}\, e^{iS_{int}[\varphi]}\, \varphi(x_1)\ldots\varphi(x_n)}{\int [d\varphi]\, e^{iS_0[\varphi]}\, e^{iS_{int}[\varphi]}}$$

$$= \frac{\int [d\varphi]\, e^{iS_0[\varphi]}\, e^{iS_{int}[\varphi]}\, \varphi(x_1)\ldots\varphi(x_n)}{\int [d\varphi]\, e^{iS_0[\varphi]}}$$

$$\times \frac{\int [d\varphi]\, e^{iS_0[\varphi]}}{\int [d\varphi]\, e^{iS_0[\varphi]}\, e^{iS_{int}[\varphi]}}$$

$$= \frac{\left\langle\!\left\langle e^{iS_{int}[\varphi]}\, \varphi(x_1)\ldots\varphi(x_n) \right\rangle\!\right\rangle}{\left\langle\!\left\langle e^{iS_{int}[\varphi]} \right\rangle\!\right\rangle}$$

in the last step we have introduced the notation

$$\left\langle\!\left\langle \mathcal{F}[\varphi] \right\rangle\!\right\rangle = \frac{\int [d\varphi]\, e^{iS_0[\varphi]}\, \mathcal{F}[\varphi]}{\int [d\varphi]\, e^{iS_0[\varphi]}}$$

<u>Gaussian mean-value</u>

perturbative expansion: $e^{iS_{int}[\varphi]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(S_{int}[\varphi]\right)^n$

translating back to operator language:

$$\langle \Omega | T\, \phi_H(x_1) \dots \phi_H(x_n) | \Omega \rangle =$$

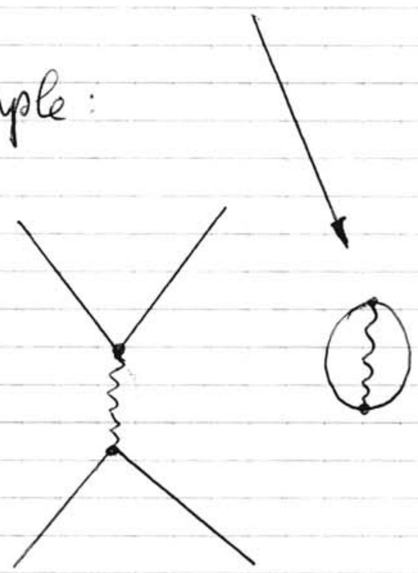
$$= \frac{\langle 0 | T\, e^{i S_{int}[\phi_{IP}]}\, \phi_{IP}(x_1) \dots \phi_{IP}(x_n) | 0 \rangle}{\langle 0 | T e^{i S_{int}[\phi_{IP}]} | 0 \rangle}$$

$|\Omega\rangle$ vacuum of full (interacting) theory

$|0\rangle$ vacuum of free theory

remark: $\langle 0 | T\, e^{i S_{int}} | 0 \rangle = e^{iL}$ is a phase factor (L is IR-divergent) which <u>cancels</u> all diagrams with <u>disconnected vacuum bubbles</u>

example:



back to fermion propagator at one-loop level:

$$\langle \Omega | T \psi_a(x) \bar{\psi}_b(y) | \Omega \rangle = \frac{\langle\!\langle e^{iS_{int}} \psi_a(x) \bar{\psi}_b(y) \rangle\!\rangle}{\langle\!\langle e^{iS_{int}} \rangle\!\rangle}$$

$$S_{int} = e \int d^4x \, \bar{\psi}(x) \not{A}(x) \psi(x) \qquad (q = -e \text{ for } e^-)$$

$$\langle\!\langle e^{iS_{int}} \psi_a(x) \bar{\psi}_b(y) \rangle\!\rangle =$$

$$= \langle\!\langle (1 + \underbrace{i S_{int}}_{\substack{\text{no contribution from} \\ \text{this term as } \langle\!\langle A_\mu \rangle\!\rangle = 0}} + \frac{i^2}{2!} S_{int}^2 + \ldots) \psi_a(x) \psi_b(y) \rangle\!\rangle$$

$$= \underbrace{\langle\!\langle \psi_a(x) \bar{\psi}_b(y) \rangle\!\rangle}_{\frac{1}{i} S_{ab}(x-y)} \quad \text{electron propagator}$$

$$+ \frac{i^2 e^2}{2!} \int d^4z_1 \, d^4z_2 \, \langle\!\langle \bar{\psi}(z_1) \gamma^\mu A_\mu(z_1) \psi(z_1)$$

$$\bar{\psi}(z_2) \gamma^\nu A_\nu(z_2) \psi(z_2) \, \psi_a(x) \bar{\psi}_b(y) \rangle\!\rangle + \ldots$$

$$= \frac{1}{i} S_{ab}(x-y)$$

$$+ \frac{i^2 e^2}{2!} \int d^4 z_1 \, d^4 z_2 \; i \overbrace{D_{\mu\nu}(z_1 - z_2)}^{\text{photon propagator}}$$

$$\langle\!\langle \overline{\Psi}_c(z_1) \gamma^\mu_{cd} \Psi_d(z_1) \overline{\Psi}_e(z_2) \gamma^\nu_{ef} \Psi_f(z_2) \Psi_a(x) \overline{\Psi}_b(y) \rangle\!\rangle$$

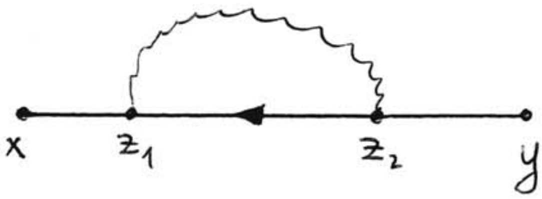
$$+ \ldots$$

$$= \frac{1}{i} S_{ab}(x-y)$$

$$+ \frac{(ie)^2}{2!} \int d^4 z_1 \, d^4 z_2 \; i D_{\mu\nu}(z_1 - z_2)$$

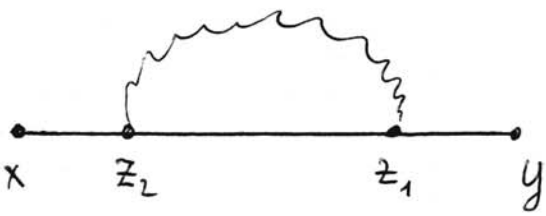
$$\left\{ \frac{1}{i} S_{ac}(x - z_1) \gamma^\mu_{cd} \frac{1}{i} S_{de}(z_1 - z_2) \right.$$

$$\gamma^\nu_{ef} \frac{1}{i} S_{fb}(z_2 - y)$$



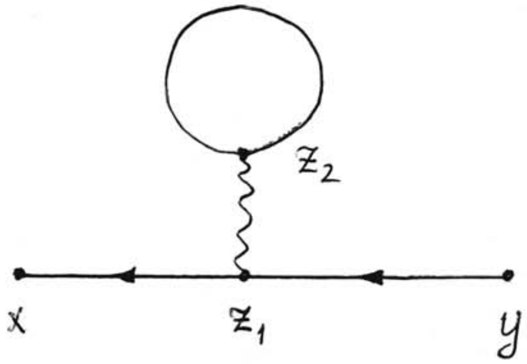
$$+ \frac{1}{i} S_{ae}(x - z_2) \gamma^\nu_{ef} \frac{1}{i} S_{fc}(z_2 - z_1)$$

$$\gamma^\mu_{cd} \frac{1}{i} S_{db}(z_1 - y)$$



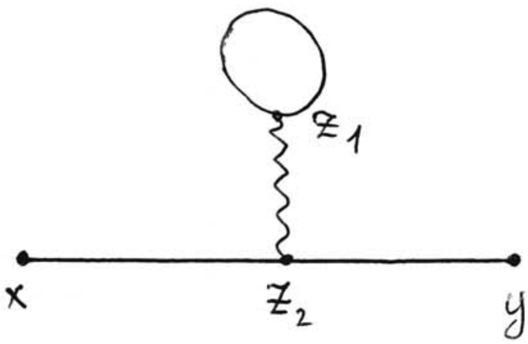
$$-\frac{1}{i} S_{ac}(x-z_1)\, \gamma^{\mu}_{cd}\, \frac{1}{i} S_{db}(z_1-y)\, \frac{1}{i} S_{fe}(0)\, \gamma^{\nu}_{ef}$$

closed fermion loop



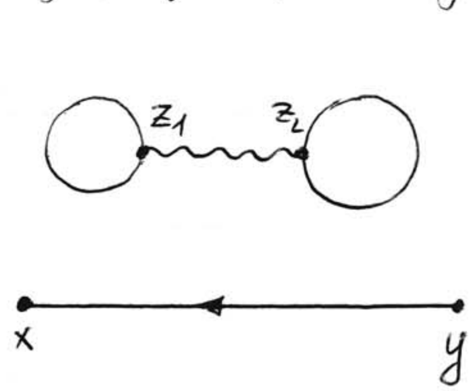
remark: S_{int} not normally ordered

$$-\frac{1}{i} S_{ae}(x-z_2)\, \gamma^{\nu}_{ef}\, \frac{1}{i} S_{fb}(z_2-y)\, \frac{1}{i} S_{dc}(0)\, \gamma^{\mu}_{cd}$$



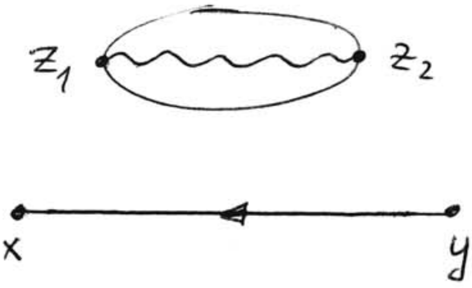
$$+\frac{1}{i} S_{ab}(x-y)\, \frac{1}{i} S_{dc}(0)\, \gamma^{\mu}_{cd}\, \frac{1}{i} S_{fe}(0)\, \gamma^{\nu}_{ef}$$

disconnected diagram
with vacuum bubble
(two-loop graph)



$$-\frac{1}{i}\, S_{ab}(x-y)\;\frac{1}{i}\, S_{fc}(z_2-z_1)\,\gamma^{\mu}_{cd}$$

$$\frac{1}{i}\, S_{de}(z_1-z_2)\,\gamma^{\nu}_{ef}\,\}$$



remarks concerning disconnected graphs with vacuum bubble:

a) <u>2-loop</u> contributions

b) first type vanishes after loop-integration

c) <u>IR-regularization</u> required

$$\int d^4 z_1\, d^4 z_2 \ldots \longrightarrow \text{finite space-time volume}$$

d) <u>cancelled</u> by $\langle\!\langle e^{iS_{int}} \rangle\!\rangle = 1 +$ 

$+$  $+ \ldots$

in the next step, I use the symmetry property of the photon propagator:

$$\langle \Omega | T \psi(x) \overline{\psi}(y) | \Omega \rangle \Big|_{\text{one-loop}} = \frac{1}{i} S(x-y)$$

$$+ \int d^4 z_1 \, d^4 z_2 \; i D^{\mu\nu}(z_1 - z_2)$$

$$\Big\{ \frac{1}{i} S(x-z_1) \, ie\gamma_\mu \, \frac{1}{i} S(z_1 - z_2) \, ie\gamma_\nu \, \frac{1}{i} S(z_2 - y)$$

$$- \frac{1}{i} S(x-z_1) \, ie\gamma_\mu \, \frac{1}{i} S(z_1 - y) \times \mathrm{Tr}\Big[\frac{1}{i} S(0) \, ie\gamma_\nu\Big] \Big\}$$

$$\mathrm{Tr}[S(0)\gamma_\nu] = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\Big[\frac{1}{m - \not{p} - i\varepsilon} \gamma_\nu\Big]$$

$$= \int \frac{d^4p}{(2\pi)^4} \, \frac{\mathrm{Tr}[(m+\not{p})\gamma_\nu]}{m^2 - p^2 - i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} \, \frac{4 p_\nu}{m^2 - p^2 - i\varepsilon}$$

expression not well-defined (requires UV-regularization; vanishes, however, in any reasonable regularization because of antisymmetric integrand)

→ remaining (relevant) expression:

$$\int d^4z_1\, d^4z_2 \;\; i\, D^{\mu\nu}(z_1 - z_2)\, \frac{1}{i}\, S(x - z_1) \;\; ie\gamma_\mu$$

$$\frac{1}{i}\, S(z_1 - z_2) \;\; ie\gamma_\nu \;\; \frac{1}{i}\, S(z_2 - y)$$

Feynman gauge

$$= \int d^4z_1\, d^4z_2 \;\; i \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(z_1 - z_2)}\, \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}$$

$$\frac{1}{i} \int \frac{d^4p_1}{(2\pi)^4}\, e^{-ip_1(x - z_1)}\, \frac{1}{m - \not{p}_1 - i\varepsilon} \;\; ie\gamma_\mu$$

$$\frac{1}{i} \int \frac{d^4p_2}{(2\pi)^4}\, e^{-ip_2(z_1 - z_2)}\, \frac{1}{m - \not{p}_2 - i\varepsilon} \;\; ie\gamma_\nu$$

$$\frac{1}{i} \int \frac{d^4p_3}{(2\pi)^4}\, e^{-ip_3(z_2 - y)}\, \frac{1}{m - \not{p}_3 - i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, \frac{d^4p_1}{(2\pi)^4}\, \frac{d^4p_2}{(2\pi)^4}\, \frac{d^4p_3}{(2\pi)^4}\, (2\pi)^4\, \delta^{(4)}(-k + p_1 - p_2)$$

$$(2\pi)^4\, \delta^{(4)}(k + p_2 - p_3)\, e^{-ip_1 x}\, e^{ip_3 y}$$

$$\frac{-ig^{\mu\nu}}{k^2 + i\varepsilon}\, \frac{i}{\not{p}_1 - m + i\varepsilon}\, ie\gamma_\mu\, \frac{i}{\not{p}_2 - m + i\varepsilon}\, ie\gamma_\nu$$

$$\frac{i}{\not{p}_3 - m + i\varepsilon}$$

$$\overset{p_2, p_3\text{-integration}}{=} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p_1}{(2\pi)^4} e^{-ip_1 x} e^{ip_1 y} \frac{-ig^{\mu\nu}}{k^2 + i\varepsilon}$$

$$\frac{i}{\not{p}_1 - m + i\varepsilon} \; ie\gamma_\mu \; \frac{i}{\not{p}_1 - \not{k} - m + i\varepsilon} \; ie\gamma_\nu \; \frac{i}{\not{p}_1 - m + i\varepsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{\not{p} - m + i\varepsilon} \; ie\gamma^\mu$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2 + i\varepsilon} \frac{i}{\not{p} - \not{k} - m + i\varepsilon}$$

$$ie\gamma_\nu \frac{i}{\not{p} - m + i\varepsilon}$$

$$\langle\Omega| T\psi(x)\bar{\psi}(y)|\Omega\rangle\Big|_{\text{one-loop}} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$$

$$\left\{ \frac{i}{\not{p} - m + i\varepsilon} + \frac{i}{\not{p} - m + i\varepsilon} \left(-i\Sigma(p)\right) \frac{i}{\not{p} - m + i\varepsilon} \right\}$$

$$-i\Sigma(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2 + i\varepsilon} \gamma^\mu \frac{1}{k^2 + i\varepsilon}$$

UV- and IR- <u>divergent</u>

UV – regularization: cut-off, Pauli-Villars,
dimensional reg.

IR – regularization: small photon mass, dim. reg.

idea of dimensional regularization:

QFT in d space-time dimensions

$$\int d^4x \to \int d^dx \qquad g^\mu{}_\mu = d$$

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}$$

choose d such that loop integral convergent

→ analytic continuation of result in variable d

treatment of γ-matrices in d dimensions:

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu}$$

$$\mathrm{Tr}\,\mathbb{1} = f(d)\,, \quad \lim_{d\to 4} f(d) = 4$$

precise form of f irrelevant, also constant f

$(f(d) = 4)$ possible

remarks: $\int d^dx\ \bar{\psi}(i\partial\!\!\!/ - m)\psi \Rightarrow 2[\psi] + 1 - d = 0$

$$\Rightarrow [\psi] = \frac{d-1}{2}$$

$$\int d^dx\ A \Box A \Rightarrow 2[A] + 2 - d = 0 \Rightarrow [A] = \frac{d-2}{2}$$

$$e\int d^dx\ \bar{\psi} A\!\!\!/ \psi \Rightarrow 0 = [e] - d + 2[\psi] + [A]$$

$$= [e] - d + d - 1 + \frac{d-2}{2}$$

$$\Rightarrow [e] = \frac{4-d}{2}$$

$$-i\Sigma(p) = -e^2 \int \frac{d^dk}{(2\pi)^d}\, \gamma_\mu \frac{p\!\!\!/ - k\!\!\!/ + m}{(p-k)^2 - m^2 + i\varepsilon}\, \gamma^\mu \frac{1}{k^2 - m_\gamma^2 + i\varepsilon}$$

$$\gamma_\mu \gamma^\mu = g_\mu{}^\mu = d$$

$$\gamma_\mu a\!\!\!/ \gamma^\mu = \gamma_\mu(-\gamma^\mu a\!\!\!/ + 2a^\mu) = (2-d)\,a\!\!\!/$$

$$-i\Sigma(p) = -e^2 \int \frac{d^dk}{(2\pi)^d} \; \frac{(2-d)(\not{p}-\not{k}) + dm}{(p-k)^2 - m^2 + i\varepsilon} \; \frac{1}{k^2 - m_\gamma^2 + i\varepsilon}$$

Feynman parametrization

$$\frac{1}{ab} = \int_0^1 dx \; \frac{1}{[ax + b(1-x)]^2}$$

$$-i\Sigma(p) = -e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(\not{p}-\not{k}) + dm}{\left\{ x\,[(k-p)^2 - m^2 + i\varepsilon] + (1-x)[k^2 - m_\gamma^2 + i\varepsilon] \right\}^2}$$

$$= -e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \; \frac{(2-d)(\not{p}-\not{k}) + dm}{[k^2 - 2x p\cdot k + x(p^2 - m^2) - (1-x)m_\gamma^2 + i\varepsilon]^2}$$

$$= -e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \; \frac{(2-d)(\not{p}-\not{k}) + dm}{[(k-xp)^2 - x^2p^2 + xp^2 - xm^2 - (1-x)m_\gamma^2 + i\varepsilon]^2}$$

new integration variable $\ell = k - xp$ (shift)

$$= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \; \frac{(2-d)(\not{p} - \overset{=0\ (\text{symm. int.})}{\not{\ell}} - x\not{p}) + dm}{[\ell^2 + x(1-x)p^2 - xm^2 - (1-x)m_\gamma^2 + i\varepsilon]^2}$$

$$= -e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(1-x)\not p + dm}{\left[k^2 + x(1-x)p^2 - xm^2 - (1-x)m_\gamma^2 + i\varepsilon\right]^2}$$

$$\left(= -e^2 \int_0^1 dy \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)\,y \not p + dm}{\left[k^2 + y(1-y)p^2 - (1-y)m^2 - y m_\gamma^2 + i\varepsilon\right]^2} \right)$$

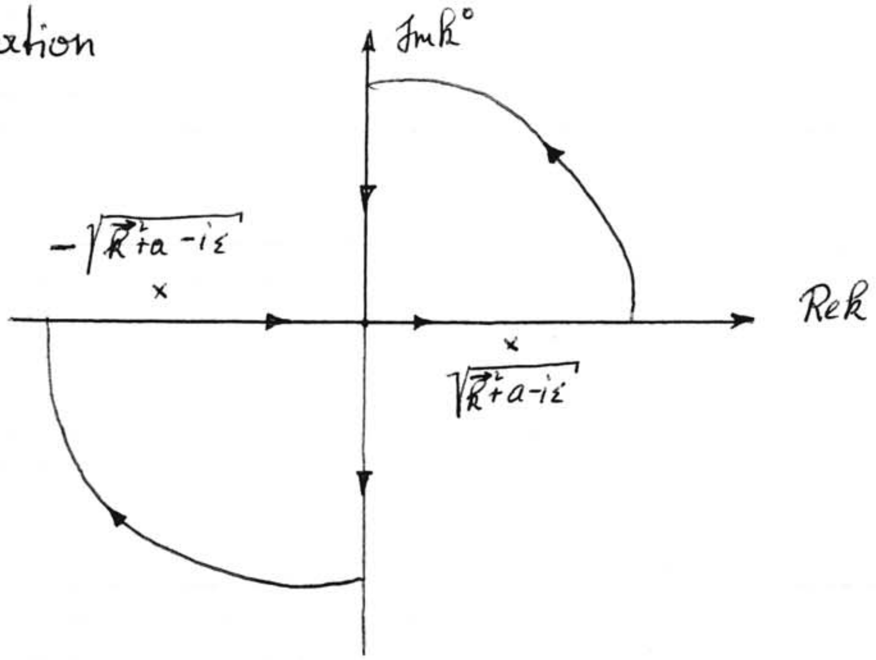
$\uparrow$

$x = 1-y$

d-dim. integral has the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - a + i\varepsilon)^\alpha} \qquad a > 0$$

Wick rotation



$$\int_{-\infty}^{+\infty} dk^0 \ldots + \int_{+i\infty}^{-i\infty} dk^0 \ldots = 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} dk^0 \ldots = \int_{-i\infty}^{+i\infty} dk^0 \ldots \underset{k^0 = i k_E^0}{=} i \int_{-\infty}^{+\infty} dk_E^0$$

$$\Rightarrow (k^0)^2 \underbrace{- (k^1)^2 - \ldots - (k^d)^2}_{-\vec{k}^2} = -(k_E^0)^2 - \vec{k}^2 =: -k_E^2$$

$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - a + i\varepsilon)^\alpha} = (-1)^\alpha i \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + a)^\alpha}$$

convergent for $d < 2\alpha$

$$\frac{1}{(k_E^2 + a)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt\, t^{\alpha - 1} e^{-t(k_E^2 + a)}$$

$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - a + i\varepsilon)^\alpha} = \frac{(-1)^\alpha i}{\Gamma(\alpha)} \int \frac{d^d k_E}{(2\pi)^d} \int_0^\infty dt\, t^{\alpha-1} e^{-t(k_E^2 + a)}$$

$$= \frac{(-1)^\alpha i}{\Gamma(\alpha)(2\pi)^d} \int_0^\infty dt\, t^{\alpha - 1} \left(\frac{\pi}{t}\right)^{d/2} e^{-ta}$$

$$= \frac{i\,(-1)^{\alpha}}{\Gamma(\alpha)\,(4\pi)^{d/2}} \int_0^\infty dt\; t^{\alpha-\frac{d}{2}-1}\, e^{-ta}$$

$$\underset{\substack{\uparrow\\ ta=s}}{=} \frac{i\,(-1)^{\alpha}}{\Gamma(\alpha)\,(4\pi)^{d/2}}\; a^{\frac{d}{2}-\alpha} \underbrace{\int_0^\infty ds\; s^{\alpha-\frac{d}{2}-1}\, e^{-s}}_{\Gamma\left(\alpha-\frac{d}{2}\right)}$$

$$\boxed{\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-a+i\varepsilon)^{\alpha}} = \frac{i\,(-1)^{\alpha}\,\Gamma\left(\alpha-\frac{d}{2}\right)}{(4\pi)^{d/2}\,\Gamma(\alpha)} (a-i\varepsilon)^{\frac{d}{2}-\alpha}}$$

<u>exercise:</u>

$$\int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^{\beta}}{(k^2-a+i\varepsilon)^{\alpha}} = \frac{(-1)^{\alpha+\beta}\, i}{(4\pi)^{d/2}} \frac{\Gamma\left(\alpha-\beta-\frac{d}{2}\right)\Gamma\left(\beta+\frac{d}{2}\right)}{\Gamma(\alpha)\,\Gamma(d/2)}$$
$$\times\; (a-i\varepsilon)^{\frac{d}{2}+\beta-\alpha}$$

α, β integer

<u>important property:</u> $\alpha=0 \Rightarrow \int \frac{d^dk}{(2\pi)^d} (k^2)^{\beta} = 0$

for arbitrary β, e.g. $\int \frac{d^dk}{(2\pi)^d} = 0$ in dim. reg.

$$\int \frac{d^dk}{(2\pi)^d}\, e^{ikx} = \delta^{(d)}(x) \Rightarrow \delta^{(d)}(0) = \int \frac{d^dk}{(2\pi)^d} = 0$$

$$-i\Sigma(p) = -e^2 \int_0^1 dx \, [\,(2-d)(1-x)\not{p} + dm\,]$$

$$\times \frac{i\,\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(2)} \left[x m^2 + (1-x) m_\gamma^2 - x(1-x)p^2 - i\varepsilon\right]^{\frac{d}{2}-2}$$

$\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$

$\Rightarrow \Sigma(p)$ has poles at $2 - \frac{d}{2} = 0, -1, -2, \ldots$

i.e. $d = 4, 6, 8, \ldots$

bevaviour of $\Sigma(p)$ in the vicinity of $d=4$:

$d = 4 - 2\varepsilon$

$$-i\Sigma(p) \xrightarrow[d\to 4]{} -e^2 \int_0^1 dx \, [\,(-2+2\varepsilon)(1-x)\not{p} +$$

$$+ (4-2\varepsilon)m\,] \frac{i\,\Gamma(\varepsilon)}{(4\pi)^2 (4\pi)^{-\varepsilon}} \left[x m^2 + (1-x)m_\gamma^2 - x(1-x)p^2 - i\varepsilon\right]^{-\varepsilon}$$

$$\varepsilon\Gamma(\varepsilon) = \Gamma(1+\varepsilon) = \underbrace{\Gamma(1)}_{1} + \varepsilon\underbrace{\Gamma'(1)}_{-\gamma_E = -0.5772} + O(\varepsilon^2)$$

(Euler constant)

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} + \Gamma'(1) + O(\varepsilon)$$

$$\Rightarrow -i\Sigma(p) = -\frac{ie^2}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \Gamma'(1) + O(\varepsilon)\right]$$

$$\times \left[1 + \varepsilon\ln(4\pi) + O(\varepsilon)\right]$$

$$\times \int_0^1 dx \left[(-2+2\varepsilon)(1-x)\not{p} + (4-2\varepsilon)m\right]$$

$$\times \left\{1 - \varepsilon\ln\left[xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2 - i\varepsilon\right] + O(\varepsilon^2)\right\}$$

$$= -\frac{ie^2}{(4\pi)^2}\left\{\left[\frac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi)\right](-\not{p} + 4m)\right.$$

$$+ \not{p} - 2m$$

$$\left. - \int_0^1 dx\left[-2(1-x)\not{p} + 4m\right]\ln\left[xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2 - i\varepsilon\right]\right\}$$

<u>interpretation of $\Sigma(p)$</u>:

loop expansion $\equiv$ expansion in powers of $\hbar$

tree propagator $\sim \hbar$

$e^{\frac{i}{\hbar}S} \longrightarrow$ vertex $\sim \frac{1}{\hbar}$

S <u>independent</u> of $\hbar$ $\rightarrow$ $m = \text{mass} \times \frac{c}{\hbar}$ = inverse Compton length!

$$\Rightarrow \Sigma \sim \underbrace{\hbar^2}_{\text{loop propagators}} \overbrace{\frac{1}{\hbar^2}}^{\text{2 vertices}}$$

$$\frac{i\hbar}{\not{p} - m} + \frac{i\hbar}{\not{p} - m}(-i\Sigma)\frac{i\hbar}{\not{p} - m} =$$

$$= \frac{i\hbar}{\not{p} - m - \hbar\Sigma} + \mathcal{O}(\hbar^3)$$

$\rightarrow$ pole of 2-point function (in momentum space) not at $\not{p} = m$ but shifted to $\not{p} = m_{ph}$

$$\Sigma(p) = A(p^2, m^2)\, \not{p} + B(p^2, m^2)$$

m (mass parameter in the Lagrangean) and m_{ph} (physical mass = pole of 2-point function) related by

$$m_{ph} - m - \underbrace{\left[A(m_{ph}^2, m^2)\, m_{ph} + B(m_{ph}^2, m^2) \right]}_{\Sigma|_{\not{p} = m_{ph}}} = 0$$

$$\Rightarrow m_{ph} = m + \underbrace{A(m^2, m^2)\, m + B(m^2, m^2)}_{\Sigma|_{\not{p}=m}} + \text{corr. of higher order in } \hbar$$

behaviour of the one-loop propagator in the vicinity of $p^2 = m_{ph}^2$:

$$\not{p} - m - \Sigma(p) = \not{p} - m - \Sigma\Big|_{\not{p}=m_{ph}} - \frac{\partial \Sigma}{\partial \not{p}}\Big|_{\not{p}=m_{ph}} (\not{p} - m_{ph})$$

$$+ \dots$$

$$= (\not{p} - m_{ph}) \left(1 - \frac{\partial \Sigma}{\partial \not{p}}\Big|_{\not{p}=m_{ph}} \right) + \mathcal{O}\left[(\not{p} - m_{ph})^2 \right]$$

$$\frac{i}{\not p - m - \Sigma(\not p)} \xrightarrow[p^2 \to m_{ph}^2]{} \frac{i\left(1 + \frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m_{ph}}\right)}{\not p - m_{ph}} =$$

$$= \frac{i Z_2}{\not p - m_{ph}}$$

$$Z_2 = 1 + \frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m_{ph}} = 1 + \frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m}$$

wave function renormalization constant (field renormalization constant) of the electron (field)

in dimensional regularization:

$$\Sigma\Big|_{\not p = m} = \frac{e^2}{(4\pi)^2}\left\{ \left[\frac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi)\right] 3m - m - \int_0^1 dx \, [2m + 2xm] \ln(x^2 m^2) \right\} =$$

$$= \frac{e^2 m}{(4\pi)^2} \Big\{ 3 \, \big[\tfrac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi) \big] - 1$$

$$- 2 \ln m^2 \int_0^1 dx \, (1+x) - 4 \int_0^1 dx \, (1+x) \ln x \Big\}$$

$$= \frac{e^2 m}{(4\pi)^2} \Big\{ 3 \, \big[\tfrac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi) - \ln m^2 \big]$$

$$+ 4 \Big\}$$

$$= \frac{e^2 m}{(4\pi)^2} \Big\{ 3 \, \big[\tfrac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi) - \ln \tfrac{m^2}{\mu^2} - \ln \mu^2 \big]$$

$$+ 4 \Big\}$$

μ is an arbitrary energy (mass) scale

$$\Rightarrow \Sigma\big|_{\not p = m} = \frac{e^2 m}{(4\pi)^2} \mu^{-2\varepsilon} \Big\{ 3 \big[\tfrac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi) -$$

$$- \ln \tfrac{m^2}{\mu^2} \big] + 4 \Big\}$$

remark: $\mu^{-2\varepsilon} \frac{1}{\varepsilon} = (1 - 2\varepsilon \ln \mu) \frac{1}{\varepsilon}$

$$= \frac{1}{\varepsilon} - \ln \mu^2$$

$[e^2] = 4 - d = 2\varepsilon$

$$\Sigma\big|_{\not p = m} = \frac{e^2 m}{(4\pi)^2}\,\mu^{d-4}\left\{3\left[\frac{2}{4-d} + \Gamma'(1) + \ln(4\pi) - \ln\frac{m^2}{\mu^2}\right] + 4\right\}$$

$$= \frac{e^2 m}{(4\pi)^2}\,\mu^{d-4}\left\{-6\,\overbrace{\left[\frac{1}{d-4} - \frac{1}{2}\left(\Gamma'(1) + \ln(4\pi)\right)\right]}^{\Lambda_d} - 3\ln\frac{m^2}{\mu^2} + 4\right\}$$

analogously one finds:

$$Z_2\big|_{\text{one-loop}} = \frac{\partial \Sigma}{\partial \not p}\bigg|_{\not p = m} =$$

$$= 1 + \frac{e^2}{(4\pi)^2}\,\mu^{d-4}\left(2\Lambda_d + \ln\frac{m^2}{\mu^2} - 4 - 2\ln\frac{m_\gamma^2}{m^2}\right)$$

IR-divergence