17. Reduction formalism

Lehmann, Symanzik, Zimmermann (LSZ)

in-states $|p_1, p_2, \ldots \text{ in}\rangle$ free particles with momenta $p_1, p_2, \ldots$ in the remote past $(t \to -\infty)$; $p_i^2 = m_{ph}^2$

out-states $|q_1, q_2, \ldots \text{ out}\rangle$ free particles with momenta $q_1, q_2, \ldots$ in the far future $(t \to +\infty)$; $q_i^2 = m_{ph}^2$

$|\Omega\rangle$ groundstate $\left(|\Omega \text{ in}\rangle = |\Omega \text{ out}\rangle\right)$

$|p\rangle$ one-particle state $\left(|p \text{ in}\rangle = |p \text{ out}\rangle\right)$

$|p_1, p_2 \text{ in}\rangle \neq |p_1, p_2 \text{ out}\rangle$ (for interacting theory)

$\langle f \text{ out} | i \text{ in}\rangle = S_{fi}$ S-matrix element

$|S_{fi}|^2$ transition probability

<u>S-operator</u> (scattering operator)

$$S\,|i\ \text{out}\rangle = |i\ \text{in}\rangle$$

$$S^\dagger S = S S^\dagger = \mathbb{1} \quad \text{unitary}$$

$$\langle f\ \text{out}|\,S\,|i\ \text{out}\rangle = \langle f\ \text{out}|i\ \text{in}\rangle = S_{fi}$$

$$\langle f\ \text{in}| = \langle f\ \text{out}|\,S^\dagger \Rightarrow \langle f\ \text{in}|\,S = \langle f\ \text{out}|$$

$$\Rightarrow \langle f\ \text{in}|\,S\,|i\ \text{in}\rangle = \langle f\ \text{out}|\,i\ \text{in}\rangle = S_{fi}$$

S_{fi} matrix-elements of S-operator in in- or out-basis

definition of <u>creation- and annihilation operators</u> for $\substack{\text{in-}\\\text{out-}}$ momentum eigenstates:

$$[a_{in}(p), a_{in}(p')^\dagger] = \delta(p, p')$$

$$a_{in}(p)\,|\Omega\rangle = 0 \qquad \forall \vec{p}$$

$$\Rightarrow \quad |p_1, p_2, \dots \text{ in} \rangle = a_{in}(p_1)^\dagger a_{in}(p_2)^\dagger \dots |\Omega\rangle$$

analogously:

$$[a_{out}(p), a_{out}(p')^\dagger] = \delta(p, p')$$

$$a_{out}(p)|\Omega\rangle = 0 \qquad \forall \vec{p}$$

$$\Rightarrow \quad |p_1, p_2, \dots, \text{out}\rangle = a_{out}(p_1)^\dagger a_{out}(p_2)^\dagger \dots |\Omega\rangle$$

we also define

$$\phi_{in}(x) = \int d\mu(p) \left(e^{-ipx} a_{in}(p) + e^{+ipx} a_{in}(p)^\dagger \right)$$

$$\phi_{out}(x) = \int d\mu(p) \left(e^{-ipx} a_{out}(p) + e^{+ipx} a_{out}(p)^\dagger \right)$$

remark: $S^\dagger a_{in}(p)\, S = a_{out}(p)$

$$\Rightarrow \quad S^\dagger \phi_{in}(x)\, S = \phi_{out}(x)$$

<u>asymptotic condition</u>

$$\lim_{x^0 \to \mp\infty} \langle \psi | \phi(x) | \chi \rangle = \sqrt{Z} \, \langle \psi | \phi_{\substack{in\\out}}(x) | \chi \rangle$$

normalizable vectors

<u>weak</u> limit $\quad \phi(x) \xrightarrow[x^0 \to \mp\infty]{} \sqrt{Z}\, \phi_{\substack{in\\out}}(x)$

(only for <u>matrix-elements</u> of the field operator)

we know:

$$a_{\substack{in\\out}}(p) = i \int d^3x \, e^{ipx} \overleftrightarrow{\partial_0} \, \phi_{\substack{in\\out}}(x)$$

$$\Rightarrow a_{\substack{in\\out}}(p)^\dagger = -i \int d^3x \, e^{-ipx} \overleftrightarrow{\partial_0} \, \phi_{\substack{in\\out}}(x)$$

the <u>normalized</u> state $|\psi\rangle = \int d\mu(p)\, \psi(p)\, |p\rangle$

is created by the operator

$$a(\psi)^\dagger = \int d\mu(p)\, \psi(p)\, a_{\substack{in\\out}}(p)^\dagger = -i \int d^3x \underbrace{\int d\mu(p)\, \psi(p)\, e^{-ipx}}_{f(x)} \overleftrightarrow{\partial_0} \, \phi_{\substack{in\\out}}(x)$$

the function $f(x) = \int d\mu(p)\, \psi(p)\, e^{-ipx}$ is a solution of the free Klein-Gordon equation $(\Box + m_{ph}^2) f(x) = 0$ with positive frequencies

remark: $[a(\psi_1), a(\psi_2)^\dagger] = \langle \psi_1 | \psi_2 \rangle$

$$= i \int d^3x\, f_1(x)^* \overleftrightarrow{\partial_0} f_2(x)$$

normalization factor in the limit $\phi(x) \xrightarrow[x^0 \to \mp\infty]{} \sqrt{Z}\, \phi_{\substack{in \\ out}}(x)$:

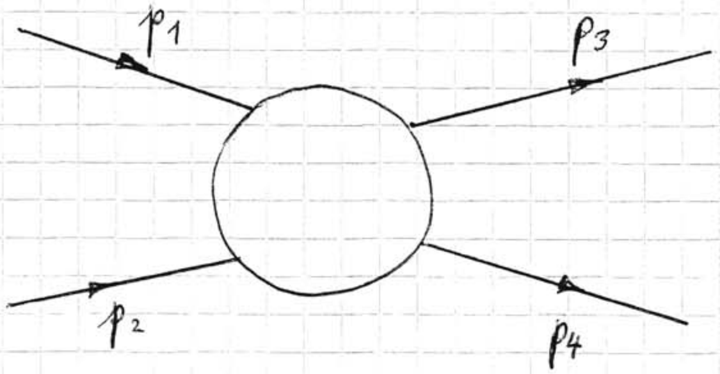
$$\langle \Omega | \phi(x) | \psi \rangle = \int d\mu(p)\, \psi(p)\, \underbrace{\langle \Omega | \phi(x) | p \rangle}_{e^{-ipx} \sqrt{Z}}$$

$$\xrightarrow[x^0 \to \mp\infty]{} \sqrt{Z}\, \langle \Omega | \phi_{\substack{in \\ out}}(x) | \psi \rangle$$

$$= \sqrt{Z} \int d\mu(p)\, \psi(p)\, \langle \Omega | \phi_{\substack{in \\ out}}(x)\, a^\dagger_{\substack{in \\ out}}(p) | \Omega \rangle$$

$$= \sqrt{Z} \int d\mu(p)\, \psi(p)\, e^{-ipx} \quad \checkmark$$

<u>reduction technique</u> for $\langle p_3, p_4 \text{ out} \mid p_1, p_2 \text{ in} \rangle$



we use <u>normalized</u> states instead of momentum eigen-"states":

$$\langle 3,4 \text{ out} \mid 1,2 \text{ in} \rangle = \langle \Omega \mid a_{3,out}\, a_{4,out}\, a^\dagger_{1,in}\, a^\dagger_{2,in} \mid \Omega \rangle$$

where $$a^\dagger_{\substack{1\\2},in} = \int d\mu(p)\, \psi_{\substack{1\\2}}(p)\, a^\dagger_{in}(p)$$

$$= -i \int d^3x\, f_{\substack{1\\2}}(x)\, \overleftrightarrow{\partial_0}\, \phi_{in}(x)$$

and $$a_{\substack{3\\4},out} = \int d\mu(p)\, \psi_{\substack{3\\4}}(p)^*\, a_{out}(p)$$

$$= i \int d^3x\, f_{\substack{3\\4}}(x)^*\, \overleftrightarrow{\partial_0}\, \phi_{out}(x)$$

$$\langle 3,4\ \text{out} \,|\, 1,2\ \text{in} \rangle = \langle 3,4\ \text{out} \,|\, a^{\dagger}_{1,\text{in}} \,|\, 2\ \text{in} \rangle$$

$$= -i \int d^3x\, f_1(x) \overleftrightarrow{\partial_0} \langle 3,4\ \text{out} \,|\, \phi_{\text{in}}(x) \,|\, 2\ \text{in} \rangle$$

$$= -\frac{i}{\sqrt{Z}} \lim_{x^0 \to -\infty} \int d^3x\, f_1(x) \overleftrightarrow{\partial_0} \langle 3,4\ \text{out} \,|\, \phi(x) \,|\, 2\text{in} \rangle$$

in the next step we use the formula

$$\int d^4x\, \partial_0\, g(x) = \lim_{x^0 \to +\infty} \int d^3x\, g(x) - \lim_{x^0 \to -\infty} \int d^3x\, g(x)$$

$$\Rightarrow \langle 3,4\ \text{out} \,|\, 1,2\ \text{in} \rangle =$$

$$= \frac{i}{\sqrt{Z}} \int d^4x\, \partial_0 \left\{ f_1(x) \overleftrightarrow{\partial_0} \langle 3,4\ \text{out} \,|\, \phi(x) \,|\, 2\ \text{in} \rangle \right\}$$

$$- \frac{i}{\sqrt{Z}} \lim_{x^0 \to +\infty} \int d^3x\, f_1(x) \overleftrightarrow{\partial_0} \langle 3,4\ \text{out} \,|\, \phi(x) \,|\, 2\ \text{in} \rangle$$

$$= \frac{i}{\sqrt{Z}} \int d^4x\, \partial_0 \left\{ f_1\, \partial_0 \langle\ \rangle - (\partial_0 f_1) \langle\ \rangle \right\}$$

$$- i \int d^3x\, f_1(x) \overleftrightarrow{\partial_0} \langle 3,4\ \text{out} \,|\, \phi_{\text{out}}(x) \,|\, 2\ \text{in} \rangle$$

$$= \frac{i}{\sqrt{Z}} \int d^4x \left\{ f_1 \partial_0^2 \langle \quad \rangle - \overbrace{(\partial_0^2 f_1)}^{(\Box + m_{ph}^2 + \Delta - m_{ph}^2) f_1} \langle \quad \rangle \right\}$$

$$+ \langle 3,4 \text{ out} \,|\, a^\dagger_{1,out} \,|\, 2 \text{ in} \rangle$$

i) $(\Box + m_{ph}^2) f_1 = 0$

ii) partial integration $\int d^3x \, (\Delta f_1) \langle \ \rangle = \int d^3x \, f_1 \Delta \langle \ \rangle$

iii) $|2 \text{ in}\rangle = |2 \text{ out}\rangle$ (one-particle state)

$$\Rightarrow \langle 3,4 \text{ out} \,|\, 1,2 \text{ in} \rangle =$$

$$= \frac{i}{\sqrt{Z}} \int d^4x \; f_1(x) \; (\Box + m_{ph}^2) \langle 3,4 \text{ out} | \phi(x) | 2 \text{ in} \rangle$$

$$+ \underbrace{\langle \Omega | a_{3,out} \, a_{4,out} \, a^\dagger_{1,out} \, a^\dagger_{2,out} | \Omega \rangle}_{\langle 3|1\rangle \langle 4|2\rangle + \langle 3|2\rangle \langle 4|1\rangle}$$

next step:

$$\langle 3,4\ \text{out} \,|\, \phi(x_1) \,|\, 2\ \text{in} \rangle =$$

$$= \langle 3,4\ \text{out} \,|\, \phi(x_1)\ a^\dagger_{2,in} \,|\Omega\rangle$$

$$= -i \int d^3x\ f_2(x)\ \overleftrightarrow{\partial_0}\ \langle 3,4\ \text{out} \,|\, \phi(x_1)\ \phi_{in}(x) \,|\Omega\rangle$$

$$= -\frac{i}{\sqrt{Z}} \lim_{x^0 \to -\infty} \int d^3x\ f_2(x)\ \overleftrightarrow{\partial_0}\ \langle 3,4\ \text{out} \,\underbrace{|\, \phi(x_1)\ \phi(x) \,|}_{\substack{T\phi(x_1)\,\phi(x) \\ (x_1^0 > x^0)}}\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}} \int d^4x\ \partial_0 \left\{ f_2(x)\ \overleftrightarrow{\partial_0}\ \langle 3,4\ \text{out} \,|\, T\phi(x_1)\ \phi(x) \,|\Omega\rangle \right\}$$

$$- \frac{i}{\sqrt{Z}} \lim_{x^0 \to +\infty} \int d^3x\ f_2(x)\ \overleftrightarrow{\partial_0}\ \langle 3,4\ \text{out} \,|\, T\phi(x_1)\,\phi(x) \,|\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}} \int d^4x\ f_2(x)\ (\Box + m_{ph}^2)\ \langle 3,4\ \text{out} \,|\, T\phi(x_1)\,\phi(x) \,|\Omega\rangle$$

$$\underbrace{- i \int d^3x\ f_2(x)\ \overleftrightarrow{\partial_0}\ \langle 3,4\ \text{out} \,|\, \phi_{out}(x)\ \phi(x_1) \,|\Omega\rangle}_{\langle 3,4\ \text{out} \,|\, a^\dagger_{2,out}\ \phi(x_1) \,|\Omega\rangle}$$

$$\langle 3,4\,\text{out}|\, a^\dagger_{2,out}\, \phi(x_1)\,|\Omega\rangle =$$

$$= \langle\Omega|\, a_{3,out}\, \underbrace{a_{4,out}\, a^\dagger_{2,out}}_{a^\dagger_{2,out}\, a_{4,out} + \langle 4|2\rangle}\, \phi(x_1)\, |\Omega\rangle$$

$$= \langle 3|2\rangle\, \langle 4|\phi(x_1)|\Omega\rangle$$

$$+ \langle 4|2\rangle\, \langle 3|\phi(x_1)|\Omega\rangle$$

$$= \langle 3|2\rangle \int d\mu(p)\, \psi_4(p)^*\, \langle p|\phi(x_1)|\Omega\rangle$$

$$+ \langle 4|2\rangle \int d\mu(p)\, \psi_3(p)^*\, \underbrace{\langle p|\phi(x_1)|\Omega\rangle}_{e^{ipx_1}\langle p|\phi(0)|\Omega\rangle}$$

this is a solution of the free Klein-Gordon equation $\rightarrow$ contribution of this term vanishes once inserted in $\langle 3,4\,\text{out}|1,2\,\text{in}\rangle =$

$$= \langle 3|1\rangle\langle 4|2\rangle + \langle 3|2\rangle\langle 4|1\rangle$$

$$+ \frac{i}{\sqrt{Z}} \int d^4x_1\, f_1(x_1)\, (\Box_1 + m_{ph}^2)\, \langle 3,4\,\text{out}|\,\phi(x_1)\,|2\,\text{in}\rangle$$

$$= \langle 3|1\rangle \langle 4|2\rangle + \langle 3|2\rangle \langle 4|1\rangle$$

$$+ \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x_1\, d^4x_2\, f_1(x_1)\, f_2(x_2)$$

$$(\Box_1 + m_{ph}^2)(\Box_2 + m_{ph}^2) \langle 3,4\ out |T\phi(x_1)\phi(x_2)|\Omega\rangle$$

now:

$$\langle 3,4\ out\ |T\phi(x_1)\phi(x_2)|\Omega\rangle =$$

$$= \langle 4\ out|\, a_{3,out}\, T\phi(x_1)\phi(x_2)\,|\Omega\rangle$$

$$= i\int d^3x\, f_3^*(x)\, \overleftrightarrow{\partial_0}\, \langle 4\, out|\, \phi_{out}(x)\, T\phi(x_1)\phi(x_2)|\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}} \lim_{x^0\to\infty} \int d^3x\, f_3^*(x)\, \overleftrightarrow{\partial_0} \langle 4\, out|\phi(x)\ T\phi(x_1)\phi(x_2)|\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}} \lim_{x^0\to\infty} \int d^3x\, f_3^*(x)\, \overleftrightarrow{\partial_0} \langle 4\, out|T\phi(x)\phi(x_1)\phi(x_2)|\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}} \lim_{x^0\to-\infty} \int d^3x\, f_3^*(x)\, \overleftrightarrow{\partial_0} \langle 4\, out|T\phi(x)\phi(x_1)\phi(x_2)|\Omega\rangle$$

$$+ \frac{i}{\sqrt{Z}} \int d^4x\ \partial_0 \left\{ f_3^*(x)\, \overleftrightarrow{\partial_0} \langle 4\, out|T\phi(x)\phi(x_1)\phi(x_2)|\Omega\rangle\right.$$

$$= i\int d^3x\, f_3^*(x)\, \overleftrightarrow{\partial_0}\, \langle 4\text{ out}|\left(T\phi(x_1)\phi(x_2)\right)\phi_{in}(x)|\Omega\rangle$$

$$+\frac{i}{\sqrt{Z}}\int d^4x\; f_3^*(x)\,(\Box + m_{ph}^2)\,\langle 4\text{ out}|T\phi(x)\phi(x_1)\phi(x_2)|\Omega\rangle$$

$$= \langle 4\text{ out}|T\phi(x_1)\phi(x_2)\underbrace{a_{3,in}|\Omega\rangle}_{0}$$

$$+\frac{i}{\sqrt{Z}}\int d^4x\; f_3^*(x)\,(\Box + m_{ph}^2)\,\langle 4\text{ out}|T\phi(x)\phi(x_1)\phi(x_2)|\Omega\rangle$$

$$\Rightarrow \quad \langle 3,4\text{ out}|1,2\text{ in}\rangle =$$

$$= \langle 3|1\rangle\langle 4|2\rangle + \langle 3|2\rangle\langle 4|1\rangle$$

$$+\left(\frac{i}{\sqrt{Z}}\right)^3\int d^4x_1\, d^4x_2\, d^4x_3\; f_1(x_1)\, f_2(x_2)\, f_3^*(x_3)$$

$$(\Box_1 + m_{ph}^2)\,(\Box_2 + m_{ph}^2)\,(\Box_3 + m_{ph}^2)$$

$$\langle 4\text{ out}|T\phi(x_1)\phi(x_2)\phi(x_3)|\Omega\rangle$$

final step:

$$\langle 4\ \text{out} \,|\, T\phi(x_1)\phi(x_2)\phi(x_3)\,|\Omega\rangle$$

$$= \langle\Omega|\, a_{4,\text{out}}\ T\phi(x_1)\phi(x_2)\phi(x_3)\,|\Omega\rangle$$

$$= i\int d^3x\ f_4^*(x)\overleftrightarrow{\partial_0}\langle\Omega|\,\phi_{\text{out}}(x)\ T\phi(x_1)\phi(x_2)\phi(x_3)\,|\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}}\lim_{x^0\to\infty}\int d^3x\, f_4^*(x)\overleftrightarrow{\partial_0}\langle\Omega|\underbrace{\phi(x)\,T\phi(x_1)\phi(x_2)\phi(x_3)}_{T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x)}|\Omega\rangle$$

$$= \frac{i}{\sqrt{Z}}\lim_{x^0\to-\infty}\int d^3x\, f_4^*(x)\overleftrightarrow{\partial_0}\langle\Omega|\underbrace{T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x)}_{(T\phi(x_1)\phi(x_2)\phi(x_3))\phi(x)}|\Omega\rangle$$

$$+ \frac{i}{\sqrt{Z}}\int d^4x\ \partial_0\left\{f_4^*(x)\overleftrightarrow{\partial_0}\langle\Omega|\,T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x)\,|\Omega\rangle\right.$$

$$= \langle\Omega|\,T\phi(x_1)\phi(x_2)\phi(x_3)\ \overbrace{a_{4,\text{in}}}^{0}\,|\Omega\rangle$$

$$+ \frac{i}{\sqrt{Z}}\int d^4x\ f_4^*(x)\left(\Box + m_{ph}^2\right)\langle\Omega|\,T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x)\,|\Omega\rangle$$

$\Rightarrow$ final result:

$\langle 3,4 \text{ out} \,|\, 1,2 \text{ in} \rangle =$

$$= \langle 3|1\rangle \langle 4|2\rangle + \langle 3|2\rangle\langle 4|1\rangle$$

$$+ \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \; f_1(x_1)\, f_2(x_2)\, f_3^*(x_3)\, f_4^*(x_4)$$

$$(\Box_1 + m_{ph}^2)\,(\Box_2 + m_{ph}^2)\,(\Box_3 + m_{ph}^2)\,(\Box_4 + m_{ph}^2)$$

$$\langle \Omega | T\, \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \,|\Omega\rangle$$

remark: in the last expression, $\langle \;\; \rangle$ can be replaced by $\langle \;\; \rangle_c$ (exercise: convince yourself that the disconnected pieces disappear when $\int d^4x\, (\Box + m_{ph}^2) \ldots$ acts on them)

back to momentum eigenstates: $\langle p_3, p_4 \text{ out} \,|\, p_1, p_2 \text{ in}\rangle =$

$$= \delta(p_1, p_3)\, \delta(p_2, p_4) + \delta(p_2, p_3)\, \delta(p_1, p_4)$$

$$+ \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1 \ldots d^4x_4 \; e^{-ip_1x_1}\, e^{-ip_2x_2}\, e^{ip_3x_3}\, e^{ip_4x_4}$$

$$(\Box_1 + m_{ph}^2) \ldots (\Box_4 + m_{ph}^2)\, \langle \Omega | T\phi(x_1) \ldots \phi(x_4) |\Omega\rangle_c$$

Fourier transform of Green function :

$$\int d^4x_1 \dots d^4x_4 \, e^{ik_1 x_1} \dots e^{ik_4 x_4} \langle \Omega | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | \Omega \rangle_c$$

$$= (2\pi)^4 \, \delta^{(4)}(k_1 + \dots + k_4) \, \Gamma_4(k_1, \dots, k_4)$$

$$\left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1 \dots d^4x_4 \, e^{-ip_1 x_1} e^{-ip_2 x_2} e^{ip_3 x_3} e^{ip_4 x_4}$$

$$(\Box_1 + m_{ph}^2) \dots (\Box_4 + m_{ph}^2) \langle \Omega | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | \Omega \rangle_c$$

$$= \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1 \dots d^4x_4 \, e^{-ip_1 x_1} e^{-ip_2 x_2} e^{ip_3 x_3} e^{ip_4 x_4}$$

$$(\Box_1 + m_{ph}^2) \dots (\Box_4 + m_{ph}^2) \int \frac{d^4x_1}{(2\pi)^4} \dots \frac{d^4x_4}{(2\pi)^4}$$

$$e^{-ik_1 x_1} \dots e^{-ik_4 x_4} \, (2\pi)^4 \, \delta^{(4)}(k_1 + \dots + k_4)$$

$$\Gamma_4(k_1, \dots, k_4)$$

$$= \int d^4x_1 \dots d^4x_1 \, e^{-ip_1x_1} e^{-ip_2x_2} e^{ip_3x_3} e^{ip_4x_4}$$

$$\int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_4}{(2\pi)^4} \, e^{-ik_1x_1} \dots e^{-ik_4x_4} \, (2\pi)^4 \delta^{(4)}(k_1 + \dots + k_4)$$

$$\underbrace{\frac{i}{\sqrt{Z}} (m_{ph}^2 - k_1^2) \dots \frac{i}{\sqrt{Z}} (m_{ph}^2 - k_4^2) \, \Gamma_4(k_1, \dots, k_4)}_{\text{amputation of external propagators in } \Gamma_4} = (*)$$

structure of Γ_4:

$$\Gamma_4(k_1, \dots, k_4) = \frac{1}{i(m_{ph}^2 - k_1^2)} \dots \frac{1}{i(m_{ph}^2 - k_4^2)} \, i R(k_1, \dots, k_4)$$

$$\Rightarrow \quad (*) = \frac{1}{Z^2} (2\pi)^d \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \, i R(-p_1, -p_2, p_3, p_4)$$

<u>on-shell</u> momenta p_1, p_2, p_3, p_4 !

$$\Rightarrow \quad \mathcal{M}(p_1, p_2 \rightarrow p_3, p_4) = \frac{1}{Z^2} R(-p_1, -p_2, p_3, p_4)$$

as $\langle p_3, p_4 \text{ out} \,|\, p_1, p_2 \text{ in} \rangle = i (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \times \mathcal{M}(p_1, p_2 \rightarrow p_3, p_4)$

general case (real scalars):

$$\langle q_1, \dots, q_m \text{ out} | k_1, \dots, k_n \text{ in} \rangle = \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \prod_{i=1}^{n} \int d^4x_i \, e^{-ik_i x_i} (\Box_{x_i} + m_{ph}^2)$$

$$\prod_{j=1}^{m} \int d^4y_j \, e^{+iq_j y_j} (\Box_{y_j} + m_{ph}^2) \langle \Omega | T \phi(y_1) \dots \phi(y_m) \phi(x_1) \dots \phi(x_n) | \Omega \rangle$$

\+ disconnected terms

spin ½ fermions

same principle, expression more complicated (spin indices, particle ≠ antiparticle)

$$\psi_{\substack{in\\out}}(x) = \sum_s \int d\mu(k) \left[b_{\substack{in\\out}}(k,s) \, e^{-ikx} u(k,s) + d^{\dagger}_{\substack{in\\out}}(k,s) \, e^{+ikx} v(k,s) \right]$$

$$b_{\substack{in\\out}}(k,s) = \int d^3x \, e^{+ikx} u^{\dagger}(k,s) \, \psi_{\substack{in\\out}}(x)$$

$$d_{\substack{in\\out}}(k,s) = \int d^3x \, \psi^{\dagger}_{\substack{in\\out}}(x) \, e^{ikx} v(k,s)$$

asymptotic condition:

$$\langle \alpha | \psi(x) | \beta \rangle \xrightarrow[x^0 \to \pm\infty]{} \sqrt{Z} \langle \alpha | \psi_{\substack{out\\in}}(x) | \beta \rangle$$

reduction (first step)

$$\langle \beta; out | \alpha, p, s; in \rangle = \frac{i}{\sqrt{Z}} \int d^4x \, \langle \beta; out | \bar{\psi}(x) | \alpha; in \rangle (i\overleftarrow{\not{\partial}} + m_{ph}) u(p,s) e^{-ipx}$$
$$+ \dots$$

$$\langle \beta; out | \alpha, \overline{p, s}; in \rangle = \frac{i}{\sqrt{Z}} \int d^4x \, e^{-ipx} \bar{v}(p,s) (i\not{\partial} - m_{ph}) \langle \beta; out | \psi(x) | \alpha; in \rangle$$
$$+ \dots$$

$$\langle \beta, p, s; out | \alpha; in \rangle = -\frac{i}{\sqrt{Z}} \int d^4x \, e^{ipx} \bar{u}(p,s) (i\not{\partial} - m_{ph}) \langle \beta; out | \psi(x) | \alpha; in \rangle$$
$$+ \dots$$

$$\langle \beta, \overline{p, s}; out | \alpha; in \rangle = -\frac{i}{\sqrt{Z}} \int d^4x \, e^{ipx} \langle \beta; out | \bar{\psi}(x) | \alpha; in \rangle (i\overleftarrow{\not{\partial}} + m_{ph}) v(p,s)$$
$$+ \dots$$

crossing symmetry: amplitude for incoming (outgoing) particle with momentum p, polarization s $\longleftrightarrow$ amplitude for outgoing (incoming) anti-particle with p, s obtained by $u(p,s)\, e^{-ipx} \leftrightarrow -v(p,s) e^{+ipx}$

complete reduction $\rightarrow$ vacuum expectation value of T-products $\rightarrow$ central objects in QFT

$$\langle \Omega | T \varphi_1(x_1) \dots \varphi_n(x_n) | \Omega \rangle$$

n-point (Green-) functions of Bose- and Fermi-fields

differential operators in reduction formula:

$$\int d^4x\, e^{ipx} (\Box + m_{ph}^2) \qquad \int d^4x\, \bar{u}(p,s)\, e^{ipx} (i\not{\partial} - m_{ph})$$

$$(m_{ph}^2 - p^2) \qquad \text{momentum space} \qquad \not{p} - m_{ph}$$

non-trivial S-matrix elements → Green functions must have poles (in momentum space) in external momenta → S-matrix elements are amputated Green functions: without poles for external particles and $p^2 = m_{ph}^2$ (on mass-shell)

Green function also defined for $p^2 \neq m_{ph}^2$ (off mass-shell) but only residues of one-particle poles are physicaly relevant ≡ S-matrix elements