

14. Spontaneous breaking of local gauge symmetries

1) Abelian gauge theory: scalar QED + scalar self interaction

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - V(\phi)$$

$$D_\mu = \partial_\mu + ieq A_\mu, \quad \phi \text{ complex scalar field}$$

$$V(\phi) = r \phi^* \phi + \lambda (\phi^* \phi)^2$$

$\lambda > 0$ (stability of the theory)

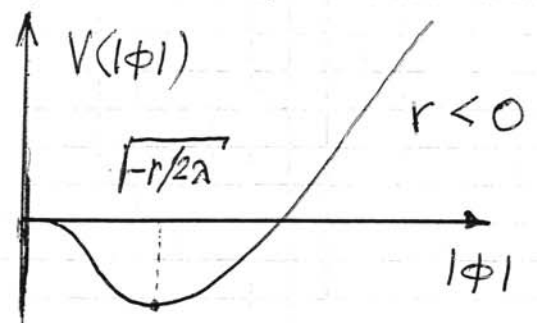
$r > 0$ no SSB $M_\phi^2 = r$ (scalar QED)

$r < 0$ SSB

gauge transformation: $\phi(x) \rightarrow e^{-iq\alpha(x)} \phi(x)$

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$$

$$V(|\phi|) = r |\phi|^2 + \lambda |\phi|^4$$



minimum of the potential determined by

$$2r|\phi| + 4\lambda|\phi|^3 = 0$$

$$\Rightarrow |\phi| (r + 2\lambda|\phi|^2) = 0$$

$$\Rightarrow |\phi|_{\min}^2 = -\frac{r}{2\lambda} = \frac{v^2}{2}, \quad v = \sqrt{-\frac{r}{\lambda}} > 0$$

$$\phi(x) = e^{i\beta(x)} \frac{1}{\sqrt{2}} (v + h(x))$$

gauge transformation $\alpha(x) = \beta(x)/q$:

$$\phi(x) \rightarrow \frac{1}{\sqrt{2}} (v + h(x)) \quad \text{"unitary gauge"}$$

$$(\mathbb{D}_\mu \phi)^* \mathbb{D}^\mu \phi \rightarrow \frac{1}{2} [\partial_\mu h - ieq A_\mu (v+h)] \times [\partial^\mu h + ieq A^\mu (v+h)] =$$

$$= \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{e^2 q^2}{2} A_\mu A^\mu (v+h)^2$$

$$= \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} \underbrace{(eqv)^2}_{m_A^2} A_\mu A^\mu$$

$$+ ve^2 q^2 A_\mu A^\mu h + \frac{1}{2} e^2 q^2 A_\mu A^\mu h^2$$

$$\mathcal{L} \xrightarrow{\text{SSB}} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu \left(1 + \frac{h}{v}\right)^2$$

$$+ \frac{1}{2} \partial_\mu h \partial^\mu h - V\left(\frac{v+h}{\sqrt{2}}\right)$$

$$V\left(\frac{v+h}{\sqrt{2}}\right) = \frac{\lambda}{2} (v+h)^2 + \frac{\lambda}{4} (v+h)^4$$

$$= -\frac{\lambda v^2}{2} (v+h)^2 + \frac{\lambda}{4} (v+h)^4$$

$$= \frac{\lambda}{2} (v+h)^2 \left[\frac{1}{2} (v+h)^2 - v^2 \right]$$

$$= \frac{\lambda}{4} (h^2 + 2vh + v^2) (h^2 + 2vh - v^2)$$

$$= \frac{\lambda}{4} \left[(2vh + h^2)^2 - v^4 \right]$$

$$= \lambda v^2 h^2 \left(1 + \frac{h}{2v}\right)^2 - \frac{\lambda v^4}{4}$$

$$M_h^2 = 2\lambda v^2$$

$$V\left(\frac{v+h}{\sqrt{2}}\right) = \frac{1}{2} M_h^2 h^2 \left(1 + \frac{h}{2v}\right)^2 + V_0$$

irrelevant constant
term

remark: physical degrees of freedom

unbroken U(1):	massless vector field	complex scalar
(r > 0)	2	2

SSB of U(1):	massive vector field	real scalar
(r < 0)	3	1

2) Nonabelian gauge theory

N real scalars (real representation of gauge group)

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix} \quad R_a := -i T_a \text{ real and antisymmetric}$$

remark: S complex $\rightarrow S = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$

$\swarrow \quad \searrow$
 real fields

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^T D^\mu \phi - V(\phi)$$

$$D_\mu \phi = (\partial_\mu + ig T_a A_\mu^a) \phi = (\partial_\mu - g R_a A_\mu^a) \phi$$

SSB $\langle 0 | \phi(x) | 0 \rangle = v \neq 0$

→ mass term for the gauge fields

$$\mathcal{L}_m(A) = \frac{1}{2} g^2 (R_a v)^T R_b v A_\mu^a A_b^\mu$$

$$M_{ab}^2 = g^2 (R_a v)^T R_b v = -g^2 v^T R_a R_b v \quad \text{mass matrix}$$

real and nonnegative as $\xi^T M^2 \xi =$

$$= \xi_a M_{ab}^2 \xi_b = g^2 (\xi_a R_a v)^T (\xi_b R_b v) \geq 0 \quad \forall \xi \in \mathbb{R}^n$$

ONB of eigenvectors of M^2 :

$$\zeta_1, \dots, \zeta_r; \quad M^2 \zeta_i = M_i^2 \zeta_i, \quad M_i^2 \neq 0$$

$$\eta_1, \dots, \eta_{n-r}; \quad M^2 \eta_R = 0$$

$$0 = \eta_R^T M^2 \eta_R = (\eta_R)_a M_{ab}^2 (\eta_R)_b =$$

$$= g^2 [(\eta_R)_a R_a v]^T (\eta_R)_b R_b v \Rightarrow (\eta_R)_a R_a v = 0$$

⇒ $\{ (\eta_R)_a R_a \mid R=1, \dots, n-r \}$ generates the unbroken subgroup $H \subset G$

the vectors $\beta_i := \frac{1}{M_i} g (\zeta_i)_a R_a v$ ($i=1, \dots, r$)

form an ONS:

$$\beta_i^T \beta_j = \frac{g^2}{M_i M_j} \sum_{a, b} (\zeta_i)_a (\zeta_j)_b (R_a v)^T R_b v$$

$$= \frac{1}{M_i M_j} \sum_{a, b} (\zeta_i)_a M_{ab}^2 (\zeta_j)_b$$

$$= \frac{1}{M_i M_j} \sum_a (\zeta_i)_a M_j^2 (\zeta_i)_a = \delta_{ij}$$

\Rightarrow the vectors $\{\beta_i\}_{i=1}^r$ are linearly independent, they span the r -dimensional space of the Goldstone bosons

remark: this implies that H is indeed a group:

$$[(\eta_R)_a R_a, (\eta_R)_b R_b] = (\eta_R)_a (\eta_R)_b f_{abc} R_c$$

$$\zeta_i \zeta_i^T + \eta_R \eta_R^T = \mathbb{1} \quad \text{completeness relation}$$

$$(\zeta_i)_a (\zeta_i)_b + (\eta_R)_a (\eta_R)_b = \delta_{ab}$$

$$\Rightarrow (\eta_R)_a (\eta_e)_b f_{abc} R_c =$$

$$= (\eta_R)_a (\eta_e)_b f_{abc} \delta_{cd} R_d$$

$$= (\eta_R)_a (\eta_e)_b f_{abc} (\zeta_i)_c (\zeta_i)_d R_d$$

$$+ (\eta_R)_a (\eta_e)_b f_{abc} (\eta_m)_c (\eta_m)_d R_d$$

to be shown: $(\eta_R)_a (\eta_e)_b f_{abc} (\zeta_i)_c = 0$

$$0 = [(\eta_R)_a R_a, (\eta_e)_b R_b] v =$$

$$= (\eta_R)_a (\eta_e)_b f_{abc} (\zeta_i)_c \underbrace{(\zeta_i)_d R_d v}_{\frac{M_i}{g} b_i}$$

the vectors $\{M_i b_i\}_{i=1}^r$ are linearly independent

$$\Rightarrow (\eta_R)_a (\eta_e)_b f_{abc} (\zeta_i)_c = 0$$

scalar mass matrix

$$(M_\phi^2)_{ij} = \left. \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right|_{\phi=v}$$

eigenvalue 0 is r -fold degenerate with
eigenvectors b_i

further $N-r$ eigenvectors denoted by c_ℓ

→ ONB $\{b_i, c_\ell\}$

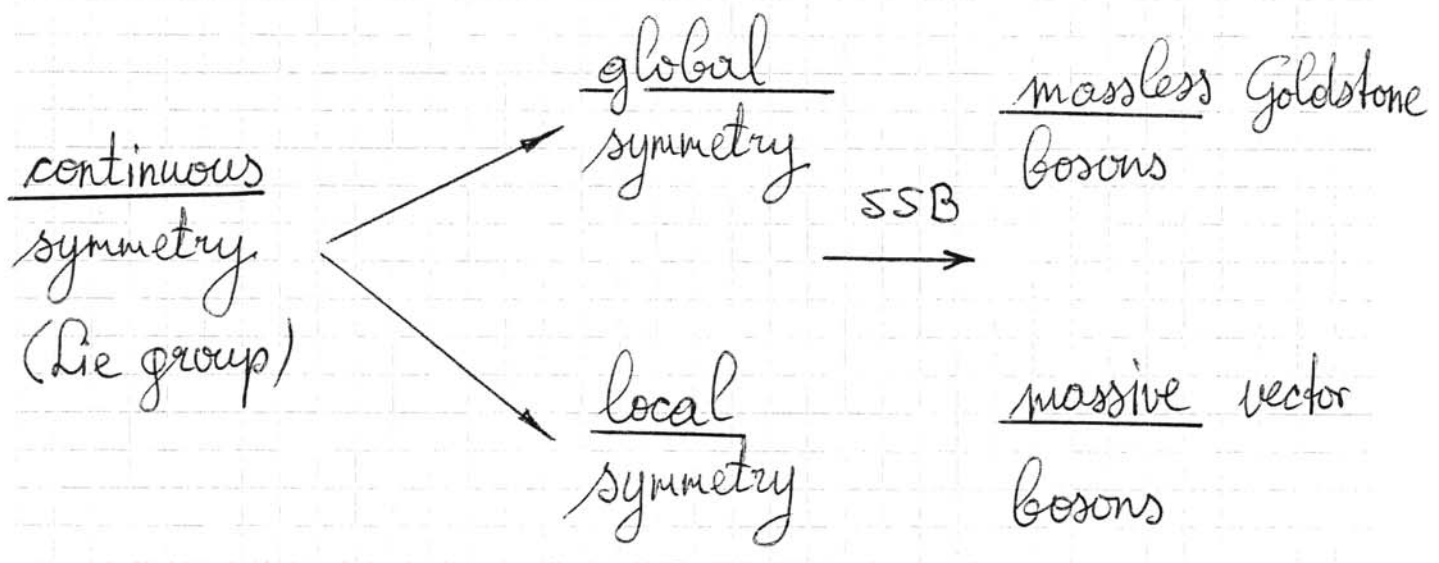
possible field parametrization

$$\begin{aligned} \phi(x) &= e^{\sum_{i=1}^r \beta_i(x) (\xi_i)_a R_a} \left(v + \sum_{\ell=1}^{N-r} c_\ell h_\ell(x) \right) \\ &= v + \sum_{\ell=1}^{N-r} c_\ell h_\ell(x) + \sum_{i=1}^r \beta_i(x) \underbrace{(\xi_i)_a R_a v}_{\frac{M_i}{g} b_i} \end{aligned}$$

gauge transformation → unitary gauge

$$\phi(x) \rightarrow v + \sum_{\ell=1}^{N-r} c_\ell h_\ell(x) \quad \text{would-be-Goldstone bosons disappear from } \mathcal{L}$$

remark:



remark: gauge theories are renormalizable ('t Hooft, Veltman), SSB does not spoil renormalizability in spite of massive vector bosons

gauge boson propagator in unitary gauge:

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}}{k^2 - M^2}$$

in "renormalizable" gauges: $\frac{k_\mu k_\nu}{M^2} \rightarrow \frac{k_\mu k_\nu}{k^2}$

+ would-be-Goldstone bosons + ghosts