13. Spontaneous symmetry breaking

symmetries in QFT:

symmetry group G of QFT, i.e. $\{U(g) | g \in G\}$ is a representation of G in the Hilbert space of state vectors with $[U(g), H] = 0 \ \forall g \in G$

Wigner-Weyl realization of a symmetry:

multiplets with respect to G

$$H |\alpha\rangle = E |\alpha\rangle$$

eigenspace associated with eigenvalue E spanned by vectors $\{|\alpha\rangle\}$

$$H U(g) |\alpha\rangle = U(g) \underbrace{H |\alpha\rangle}_{E|\alpha\rangle} = E\, U(g) |\alpha\rangle$$

$$U(g) |\alpha\rangle = \sum_\beta |\beta\rangle \underbrace{\langle\beta| U(g) |\alpha\rangle}_{U_{\beta\alpha}(g)}$$

$\Rightarrow$ vectors $\{|\alpha\rangle\}$ form an invariant subspace under G (multiplet)

<u>remark:</u> this situation is well known from QM

potential invariant under rotations $\rightarrow$ angular momentum conserved $\rightarrow$ degenerate energy levels for given total angular momentum

<u>if</u> ground state unique $\Rightarrow U(g)|0\rangle = |0\rangle$ $\forall\, g \in G$ (ground state invariant under G)

basic assumption in QFT: unique vacuum state $|0\rangle$ ($P^\mu |0\rangle = 0$)

<u>Nambu-Goldstone</u> realization of a symmetry: ground state <u>not</u> invariant under the symmetry group of the Hamiltonian

examples from solid state physics:

ferromagnet: Hamiltonian invariant under rotations
but: for $T < T_c$ (Curie temperature) ground state not rotation invariant; no orientation of the spins energetically preferred, but a specific one singled out in the ground state $\rightarrow$ rotational symmetry spontaneously broken

infinite crystalline solid: translational symmetry spontaneously broken

assumption: theory (Hamiltonian) invariant under Lie group G (continuous group)

$$\partial_\mu J^\mu(x) = 0 \quad \rightarrow \quad Q = \int d^3x \, J^0(x)$$

$$\| Q|0\rangle \|^2 = \langle 0| Q\, Q|0\rangle =$$

$$= \int d^3x \, \langle 0| Q\, J^0(x) |0\rangle$$

$$= \int d^3x \, \langle 0| Q\, e^{iPx} J^0(0)\, e^{-iPx} |0\rangle$$

$$= \int d^3x \, \underbrace{\langle 0| Q\, J^0(0) |0\rangle}_{\text{independent of } x}$$

$$\Downarrow$$

Goldstone alternative

$Q|0\rangle = 0$	$\| Q|0\rangle \| = \infty$
Wigner – Weyl	(i.e. Q not defined)
linear representation of the symmetry group	Nambu – Goldstone
degenerate multiplets	nonlinear realization of the symmetry
unbroken (exact) symmetry	massless Goldstone bosons
	SSB

more precisely:

$$\lim_{V\to\infty} \int_V d^3x\, [H, J_a^0(x)] = 0, \text{ but } \lim_{V\to\infty} \left\| \int_V d^3x\, J_a^0(x)\, |0\rangle \right\| = \infty$$

<u>Goldstone theorem</u> (important consequence of SSB)

$$Q_a^V(x^0) = \int_V d^3x\, J_a^0(x) \quad \text{time-dependent in finite volume } V$$

<u>basic assumption</u>: $\exists$ local operator $\mathcal{O}(0)$ transforming under G in a <u>nontrivial</u> manner, with

$$\lim_{V\to\infty} \overbrace{\langle 0 | [Q_a^V(x^0), \mathcal{O}(0)] | 0\rangle}^{f_a^V(x^0)} \neq 0$$

(only possible for $Q_a |0\rangle \neq 0$)

standard example in particle physics:

scalar field $\phi_m(x)$: $\quad [Q_a, \phi_m(x)] = -(T_a)_{mn}\, \phi_n(x)$

$\langle 0 | \phi_n(x) | 0 \rangle \neq 0 \rightarrow$ SSB

$\lim_{V\to\infty} f_a^V(x^0)$: <u>order parameter</u> of SSB

$$f_a^V(x^0) = \int_V d^3x \sum_n \{ \langle 0 | J_a^0(x) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle$$
$$- \langle 0 | \mathcal{O}(0) | n \rangle \langle n | J_a^0(x) | 0 \rangle \}$$

$$\langle 0 | J_a^0(x) | n \rangle = \langle 0 | e^{iPx} J_a^0(0) e^{-iPx} | n \rangle =$$
$$= e^{-ip_n x} \langle 0 | J_a^0(0) | n \rangle$$

$$\Rightarrow f_a^V(x^0) = \int_V d^3x \sum_n \{ \langle 0 | J_a^0(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle e^{-ip_n x}$$
$$- \langle 0 | \mathcal{O}(0) | n \rangle \langle n | J_a^0(0) | 0 \rangle e^{+ip_n x} \}$$

$$\frac{df_a^V(x^0)}{dx^0} = \int d^3x \sum_n \{ -ip_n^0 \langle 0 | J_a^0(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle e^{-ip_n x}$$
$$- ip_n^0 \langle 0 | \mathcal{O}(0) | n \rangle \langle n | J_a^0(0) | 0 \rangle e^{+ip_n x} \}$$

on the other hand:

$$\frac{df_a^V(x^0)}{dx^0} = \int_V d^3x \langle 0 | \left[\frac{\partial J_a(x)}{\partial x^0}, \mathcal{O}(0) \right] | 0 \rangle$$

$$= -\int_V d^3x \, \langle 0 | [\vec{\nabla} \vec{J}_a(x), \mathcal{O}(0)] | 0 \rangle =$$

$$= -\int_{\partial V} d\vec{f} \, \langle 0 | [\vec{J}_a(x), \mathcal{O}(0)] | 0 \rangle \xrightarrow[V \to \infty]{} 0$$

$$\Rightarrow \lim_{V \to \infty} \frac{df_a^V(x^0)}{dx^0} = 0 \text{ , but } \lim_{V \to \infty} f_a^V(x^0) \neq 0$$

i.e.

$$0 \neq (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \{ \langle 0 | J_a^0(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle e^{-ip_n^0 x^0}$$
$$- \langle 0 | \mathcal{O}(0) | n \rangle \langle n | J_a^0(0) | 0 \rangle e^{+ip_n^0 x^0} \}$$

$$0 = (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \{ \langle 0 | J_a^0(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle e^{-ip_n^0 x^0}$$
$$+ \langle 0 | \mathcal{O}(0) | n \rangle \langle n | J_a^0(0) | 0 \rangle e^{+ip_n^0 x^0} \}$$

für arbitrary $x^0 \Rightarrow \exists$ state $|n\rangle$ with

$$\langle 0 | J_a^0(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle \neq 0 \text{ , } p_n^0 = 0 \text{ for } \vec{p}_n = 0$$

$\rightarrow$ <u>massless state</u>

$$p_n^0 = \sqrt{m^2 + \vec{p}_n^{\,2}} \rightarrow m = 0$$

Goldstone theorem:

spontaneously broken symmetry → massless states with same quantum numbers as

$$J_a^0(0)|0\rangle\,,\quad \mathcal{O}(0)|0\rangle$$

most symmetries: J_a^0 rotation-inv. bosonic operator → $|n\rangle$ spin 0 state (Goldstone boson)

exception: SUSY → fermionic current → Goldstone fermion possible (in principle)

remarks:

a) SSB of discrete symmetries (like P, CP) does not give rise to Goldstone bosons

b) SSB of local gauge symmetries → Goldstone bosons disappear from the particle spectrum → gauge fields become massive (Higgs-Kibble mechanism): superconductor (massive photon), electroweak interaction (massive $W^{\pm}, Z^0$ bosons) manifestation of Goldstone theorem in <u>solid-state</u> physics:

relativistic relation $p_n^0 = \sqrt{m^2 + \vec{p}^{\,2}}$

solid-state: dispersion relation $\omega(k)$ $(k = 2\pi/\lambda)$

existence of Goldstone modes $\rightarrow$ $\lim\limits_{k\to 0} \omega(k) = 0$

crystal: longitudinal <u>phonons</u>

ferromagnet: spin-waves (<u>magnons</u>)

<u>example for SSB: vacuum expectation value of a scalar field</u>

order parameter $\langle 0 | \phi(x) | 0 \rangle \neq 0$ for scalar field $\phi(x)$

we consider the following <u>model</u>:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi - V(\phi)$$

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \quad N \text{ } \underline{\text{real}} \text{ scalar fields}$$

$\mathcal{L}$ invariant under Lie group G with representation $e^{-i\alpha_a T_a}$ acting on ϕ (restricts potential $V(\phi)$)

remark: $T_a{}^T = -T_a = T_a{}^*$ (ϕ real)

Noether current: $J_a^\mu = \frac{i}{2} \phi^T T_a \overleftrightarrow{\partial^\mu} \phi$

$$Q_a = \int d^3x \, J_a^0$$

infinitesimal form of $e^{i\alpha . Q} \phi \, e^{-i\alpha . Q} = e^{-i\alpha . T} \phi$:

$$[Q_a, \phi_m] = -(T_a)_{mn} \phi_n$$

we assume: $\langle 0 | \phi_m(x) | 0 \rangle = v_m$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \neq 0$$

$$\langle 0 | [Q_a, \phi_m] | 0 \rangle = -(T_a)_{mn} v_n$$

explicit determination of the Goldstone bosons of our scalar model (tree approximation)

a) vacuum expectation value (in the tree approximation) determined by the minimum of $V(\phi)$:

$$\left.\frac{\partial V}{\partial \phi_m}\right|_{\phi=v} = 0 \quad \forall m$$

b) mass matrix

$$V(\phi) = V(v) + \frac{1}{2!} \left.\frac{\partial^2 V}{\partial \phi_m \partial \phi_n}\right|_{\phi=v} (\phi - v)_m (\phi - v)_n + \dots$$

$$\phi' := \phi - v$$

$\rightarrow$ mass matrix $\quad (M^2)_{mn} = \left.\frac{\partial^2 V}{\partial \phi_m \partial \phi_n}\right|_{\phi=v}$

c) invariance of $V(\phi)$

$$V(\phi) = V\left(e^{-i\alpha_a T_a}\phi\right)$$

$$= V(\phi) - i\,\alpha_a (T_a\phi)_n \frac{\partial V}{\partial\phi_n} + \dots$$

$$\Rightarrow \quad \frac{\partial V}{\partial\phi_n} (T_a\phi)_n = 0$$

d) massless degrees of freedom

differentiate previous equation $\left(\frac{\partial}{\partial\phi_m}\dots\right)$

$$\rightarrow \quad \frac{\partial^2 V}{\partial\phi_m \partial\phi_n} (T_a\phi)_n + \frac{\partial V}{\partial\phi_n} (T_a)_{nm} = 0$$

insert $\phi = v \rightarrow \quad (M^2)_{mn} (T_a v)_n = 0 \;\; \forall a$

e) number of Goldstone bosons

dimension of space spanned by $\{T_a v\}_a$

= # of Goldstone bosons = "# of generators with $T_a v \neq 0$"

$\{b_1, \dots, b_{N_\pi}\}$ ONB of $\langle\!\langle \{T_a v\}_a \rangle\!\rangle$

$$\Rightarrow \quad \pi_r = b_r^T \phi' , \quad r = 1, \dots, N_\pi = \dim(G) - \dim(H)$$

f) unbroken subgroup H generated by those generators $T = \sum_a \alpha_a T_a$ with

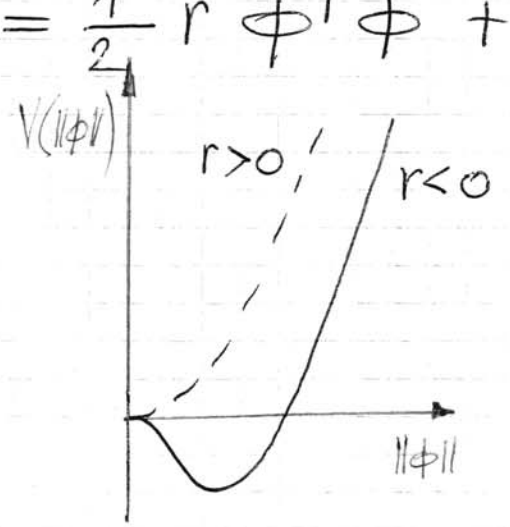
$$T v = 0$$

<u>example:</u> $G = SO(N)$, $N \geq 2$

$$\dim(G) = \frac{N(N-1)}{2} = \binom{N}{2}$$

ϕ transforming according N-dim. (defining) repr. of $SO(N)$ (<u>linear σ-model</u>)

$$V(\phi) = \frac{1}{2} r \phi^T \phi + \frac{\lambda}{4} (\phi^T \phi)^2$$



$r > 0$ <u>no</u> SSB

$\rightarrow$ N scalars with $M^2 = r$

<u>$r < 0$ SSB</u>

$$\|\phi\| = \sqrt{\phi^T \phi}$$

$$V = \frac{r}{2} \|\phi\|^2 + \frac{\lambda}{4} \|\phi\|^4$$

$$\frac{\partial V}{\partial \|\phi\|} = r \|\phi\| + \lambda \|\phi\|^3 = \|\phi\| \left(r + \lambda \|\phi\|^2 \right) = 0$$

$r < 0 \rightarrow$ minimum at $\|\phi\|^2 = -\frac{r}{\lambda}$

choose $v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \|v\| \end{bmatrix}$; $\|v\| = \sqrt{-\frac{r}{\lambda}}$

$H = SO(N-1)$ unbroken subgroup

$$\Rightarrow N_\pi = \dim(G) - \dim(H) =$$

$$= \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2}$$

$= N - 1$ Goldstone bosons

$\Rightarrow \exists$ N-1 generators which do <u>not</u> annihilate v :

$$iT_1 = \left[\begin{array}{ccc|c} & & & 1 \\ & 0 & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline -1 & 0 \dots & 0 & 0 \end{array}\right], \quad iT_2 = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & 1 \\ & & 0 & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline 0 & -1 & 0 \dots & 0 & 0 \end{array}\right], \dots$$

$$\langle\langle \{ iT_a v \mid a = 1, \dots, N-1 \} \rangle\rangle =$$

$$= \langle\langle \{ e_1, \dots, e_{N-1} \} \rangle\rangle$$

$\pi_1 = \phi_1, \dots, \pi_{N-1} = \phi_{N-1}$ massless Goldstone fields

only $\phi_N' = \phi_N - \|v\|$ is massive

check by inspection of $V(\phi)$:

$$V(\phi) = V(v + \phi') =$$

$$= \frac{1}{2} r \left[\phi_1^2 + \dots + \phi_{N-1}^2 + (\phi_N' + \|v\|)^2 \right]$$

$$+ \frac{\lambda}{4} \left[\qquad -\!\!/\!/\!\!- \qquad \right]^2 =$$

$$= \frac{r}{2} \left[\phi_1^2 + \dots + \phi_{N-1}^2 + (\phi_N' + \|v\|)^2\right]$$

$$+ \frac{\lambda}{4} \left[\phi_1^2 + \dots + \phi_{N-1}^2\right]^2$$

$$+ \frac{\lambda}{2} \left[\phi_1^2 + \dots + \phi_{N-1}^2\right] (\phi_N' + \|v\|)^2$$

$$+ \frac{\lambda}{4} (\phi_N' + \|v\|)^4$$

$$= \underbrace{\frac{r}{2} \|v\|^2 + \frac{\lambda}{4} \|v\|^4}_{V(v) = const.}$$

$$+ \underbrace{(r \|v\| + \lambda \|v\|^3)}_{0} \phi_N'$$

$$+ \frac{1}{2} \underbrace{(r + \lambda \|v\|^2)}_{0} (\phi_1^2 + \dots + \phi_{N-1}^2)$$

$$+ \frac{1}{2} \underbrace{(r + 3\lambda \|v\|^2)}_{2\lambda \|v\|^2 = M_N^2} (\phi_N')^2$$

+ cubic and quartic interaction terms

<u>example</u>: SSB of <u>chiral symmetry</u> in QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^{\alpha}_{\mu\nu} G_{\alpha}^{\mu\nu} + \sum_f (\overline{q}_f i\gamma^\mu D_\mu q_f - m_f \overline{q}_f q_f)$$

$f = u, d, c, s, t, b$

quark <u>mass hierarchy</u> in QCD:

$$m_u, m_d \ll m_s \ll m_c, m_b, m_t$$

$\rightarrow$ study limit of

i) $N_f = 2$ $\qquad\qquad\qquad\qquad$ $m_u = m_d = 0$

massless quark flavours

ii) $N_f = 3$ $\qquad\qquad\qquad\qquad$ $m_u = m_d = m_s = 0$

<u>chiral limit</u>

$$\mathcal{L}^{(0)}_{QCD} = \overline{q}_L i\gamma^\mu D_\mu q_L + \overline{q}_R i\gamma^\mu D_\mu q_R$$

$$- \frac{1}{4} G^{\alpha}_{\mu\nu} G_{\alpha}^{\mu\nu} + \mathcal{L}_{\text{heavy quarks}}$$

$$q = \begin{pmatrix} u \\ d \\ (s) \end{pmatrix}, \quad q_{L,R} = \frac{1 \mp \gamma_5}{2} q$$

$\mathcal{L}_{QCD}^{(0)}$ is invariant under the global chiral transformations

$$q_R \to V_R\, q_R\,,\quad q_L \to V_L\, q_L$$

$$V_{R,L} \in U(N_f)$$

Noether currents:

$$V_i^\mu = \bar{q}\gamma^\mu \frac{\lambda_i}{2} q\,,\quad A_i^\mu = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda_i}{2} q \qquad i=1,\dots,N^2-1$$

$$V_0^\mu = \bar{q}\gamma^\mu q\,,\quad A_0^\mu = \bar{q}\gamma^\mu\gamma_5 q$$

$U(1)_A$ anomaly: $\partial_\mu A_0^\mu = \frac{N_f \alpha_s}{4\pi} G_{\mu\nu}^\alpha \tilde{G}_\alpha^{\mu\nu}$

$\rightarrow$ actual symmetry group of massless QCD:

$$\underbrace{SU(N_f)_L \times SU(N_f)_R}_{\text{chiral group}} \times U(1)_V$$

chiral group (chiral symmetry)

$U(1)_V$: $\int d^3x\, V_0^0(x)$ counts #quarks − #antiquarks

$\rightarrow$ $3\,V_0^\mu$ baryon number current

<u>real world</u>: chiral symmetry <u>explicitly</u> broken by <u>quark mass term</u> $\overline{q_L} \mathcal{M} q_R + \overline{q_R} \mathcal{M} q_L$

$\mathcal{M} = \mathrm{diag}\,(m_u, m_d, (m_s))$ mass-matrix of light quarks

QCD vacuum $\longrightarrow$ chiral symmetry <u>spontaneously broken</u>

$$SU(N_f)_L \times SU(N_f)_R \longrightarrow SU(N_f)_V$$

order parameters: non-vanishing <u>quark condensates</u>

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle \left(= \langle 0|\bar{s}s|0\rangle\right) \neq 0$$

vectorial subgroup $SU(N_f)_V$ remains <u>unbroken</u> (symmetry of the vacuum)

# of GBs = # of broken generators

$N_f = 2$: 3 GBs $\pi^\pm, \pi^0$

$N_f = 3$: 8 GBs $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

<u>pseudoscalar</u> mesons

in the real world $m_{u,d,s} \neq 0$ → chiral symmetry also explicitly broken → pseudoscalars become massive

→ effects of light quark masses can be treated perturbatively