12. Nonabelian gauge theories

reminder: QED is abelian gauge theory
(gauge group $U(1)$)

QED of spin ½ fields $\psi(x)$ and complex scalar fields $\phi(x)$

e.g. $\psi = \begin{bmatrix} e \\ \mu \\ \tau \\ u_r \\ u_g \\ u_b \\ \vdots \end{bmatrix}$, $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{bmatrix}$

mass matrix

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$$
$$+ (D_\mu \phi)^\dagger D^\mu \phi - V(\phi)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

charge matrix of spin ½ fields

$$D_\mu \psi = (\partial_\mu + ie Q A_\mu)\psi$$

$$D_\mu \phi = (\partial_\mu + ie Q' A_\mu)\phi$$

charge matrix of spin 0 fields

$\mathcal{L}_{QED}$ is invariant under the <u>local</u> $U(1)$ gauge transformation

$$\psi(x) \rightarrow e^{-iQ\alpha(x)}\,\psi(x)$$

$$\phi(x) \rightarrow e^{-iQ'\alpha(x)}\,\phi(x)$$

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e}\,\partial_\mu \alpha(x)$$

- <u>generalization</u>: <u>nonabelian</u> gauge group (Yang - Mills 1954)

compact <u>Lie group</u> G

$$U(\alpha_1, \dots, \alpha_n) = e^{-i\alpha_a T_a} \qquad \alpha_a \in \mathbb{R}$$

$T_a^\dagger = T_a$, $a = 1, \dots, n =: d(G)$ = number of <u>generators</u>

<u>Lie algebra</u> $\mathcal{L}$ (of G) spanned by all real linear combinations of $T_1, \dots, T_n$

commutation relation

$$[T_a, T_b] = i \underbrace{f_{abc}}_{\text{structure constants}} T_c$$

usual choice: orthogonal basis $\mathrm{Tr}(T_a T_b) = c\,\delta_{ab}$

$\Rightarrow$ f_{abc} totally antisymmetric (exercise)

(linear) representation of Lie algebra $\mathcal{L}$:

$T_a \to D(T_a)$, $D(T_a)$ are linear operators

with $[D(T_a), D(T_b)] = i f_{abc} D(T_c)$

adjoint representation: the matrices $(t_a)_{bc} = -i f_{abc}$ are building an n-dimensional representation of the Lie algebra $\mathcal{L}$ (exercise)

examples:

1) $SU(2)$ $\qquad U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}$

three real parameters $\alpha_1, \alpha_2, \alpha_3$

Pauli matrices $\sigma_1, \sigma_2, \sigma_3$

$T_a = \frac{\sigma_a}{2}$ generators of $SU(2)$

Lie algebra $su(2)$ of $SU(2)$ characterized by fundamental commutation relations

$$[T_a, T_b] = i\, \varepsilon_{abc}\, T_c$$

with structure constants $\varepsilon_{abc} = \begin{cases} +1 & a,b,c \text{ \underline{even} perm. of } 1,2,3 \\ -1 & \text{—''— \underline{odd} —''— —''—} \\ 0 & \text{otherwise} \end{cases}$

of su(2)

normalization: $\mathrm{Tr}\,(T_a T_b) = \frac{1}{2}\delta_{ab}$

adjoint representation: three 3×3 matrices $(t_a)_{bc} = -i\,\varepsilon_{abc}$

$[t_a, t_b] = i\,\varepsilon_{abc}\, t_c$

2) $SU(3)$ = group of <u>unitary</u> 3×3 matrices U with $\det U = 1$

Lie algebra $su(3) = \{ T \mid T^\dagger = T \wedge \mathrm{Tr}\, T = 0 \}$

$\Rightarrow \dim su(3) = 8 = d(SU(3))$

defining representation of $su(3)$:

basis $T_a = \frac{\lambda_a}{2}$

Gell-Mann matrices $\lambda_1, \dots, \lambda_8$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

<u>construction of a nonabelian gauge theory</u>

$$\mathcal{L}_\psi = i \bar{\psi} \gamma^\mu (\partial_\mu + i g T_a A_\mu^a) \psi - \bar{\psi} m \psi$$

ψ fermionic multiplet with respect to gauge group G with representation $U = e^{-i\alpha_a T_a}$

$[T_a, m] = 0$ restricts mass matrix m

analogously:

$$\mathcal{L}_\phi = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi)$$

for scalar multiplet ϕ

$$D_\mu \phi = (\partial_\mu + ig T_a' A_\mu^a) \phi$$

<u>example</u>: <u>QCD</u> $G = SU(3)$

each quark flavour (u, d, c, s, t, b) has three "colour" degrees of freedom, e.g.

$$u(x) = \begin{bmatrix} u_r(x) \\ u_g(x) \\ u_b(x) \end{bmatrix}$$ transform according to defining repr. of $SU(3)$

$$\mathcal{L}_{QCD} = \sum_q \left\{ \bar{q}(x)\, i\gamma^\mu \left(\partial_\mu + i g_s \frac{\lambda_a}{2} G^a_\mu(x)\right) q(x) - m_q\, \bar{q}(x)\, q(x) \right\} + \mathcal{L}_G$$

$q = u, d, \ldots$

gluon field $G^a_\mu(x)$, $a = 1, \ldots, 8$

back to the <u>general case</u>:

<u>local</u> gauge transformation

$$\psi(x) \to U(x)\,\psi(x)$$

$$U(x) = e^{-i\alpha_a(x) T_a}$$

$$T_a A^a_\mu(x) =: A_\mu(x) \longrightarrow A'_\mu(x)$$

condition for invariance of $\mathcal{L}_\psi$ under local gauge transformation

$$D_\mu \psi = (\partial_\mu + i g A_\mu)\psi \to (\partial_\mu + i g A'_\mu)\, U \psi$$

$$= U \partial_\mu \psi + (\partial_\mu U)\psi + ig A'_\mu U \psi$$

$$= U [\partial_\mu \quad + ig(U^{-1} \; A'_\mu U - \frac{i}{g} U^{-1} \partial_\mu U)] \psi$$

$$= \; U D_\mu \psi$$

$\uparrow$

<u>if</u> $\quad U^{-1} A'_\mu U - \frac{i}{g} U^{-1} \partial_\mu U = A_\mu$

$$\Updownarrow$$

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} \; (\partial_\mu U) U^{-1}$$

A'_μ can be written in the form $A'_\mu = T_a A'^a_\mu$

<u>proof:</u>

we consider the expression $X(t) := e^{-itY} X e^{itY}$

X, Y are quadratic matrices

$X(t)$ fulfils the differential equation

$\dot{X}(t) = -i\,[Y, X(t)]$

with the initial condition $X(0) = X$

I define the linear operator

$$(\mathrm{ad}\,Y)\,X := [Y, X]$$

acting on the space of matrices;

with this definition, the differential equation can be written in the form

$$\dot{X}(t) = -i\,(\mathrm{ad}\,Y)\,X(t)\,,$$

the solution is given by

$$X(t) = e^{-it\,\mathrm{ad}\,Y}\,\underbrace{X(0)}_{X}$$

$$\Rightarrow\; e^{-iY} X e^{iY} = X(1) = e^{-i\,\mathrm{ad}\,Y} X$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\mathrm{ad}\,Y)^n\, X$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \underbrace{[Y, [Y, \ldots [Y, X] \ldots]}_{n\text{-fold commutator}}$$

we apply this formula to our problem:

$$X = T_a , \quad Y = \alpha_b T_b =: \alpha \cdot T$$

$$e^{-i\alpha.T} T_a e^{i\alpha.T} = e^{-i\,ad(\alpha.T)} T_a =$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [\alpha.T, [\alpha.T, \dots [\alpha.T, T_a] \dots]$$

$$(ad\,\alpha.T)\, T_a = [\alpha_b T_b , T_a] = \alpha_b\, i f_{bac} T_c$$

$$= T_c\, \alpha_b \underbrace{(-i f_{bca})}_{(t_b)_{ca}} = T_c\, \alpha_b (t_b)_{ca}$$

$$\Rightarrow \quad e^{-i\alpha.T} T_a e^{i\alpha.T} = T_c \left(e^{-i\alpha_b t_b}\right)_{ca}$$

$$\Rightarrow \quad U A_\mu U^{-1} = e^{-i\alpha.T} T_a A^a_\mu e^{i\alpha.T}$$

$$= T_c \left(e^{-i\alpha_b t_b}\right)_{ca} A^a_\mu$$

second term:

$$U(x + t e(\mu))\, U(x)^{-1} = e^{-i\alpha_a(x+te(\mu))T_a}\, e^{i\alpha_b(x)T_b} =$$

unit vector in μ-direction

$$= e^{-i\beta_a(x,t)T_a}$$

$$U(x)\,U(x)^{-1} = \mathbb{1} \Rightarrow \beta_a(x,0) = 0$$

$$\frac{d}{dt} U(x + t e_{(\mu)})\, U(x)^{-1} \Big|_{t=0} = \left(\partial_\mu U(x)\right) U(x)^{-1}$$

$$= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\beta_a(x,t)\, T_a\right)^n \Big|_{t=0} =$$

$$= -i \, \frac{\partial \beta_a(x,t)}{\partial t} \Big|_{t=0} T_a$$

$$\Rightarrow \quad A'_\mu = U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U)\, U^{-1}$$

can be written as $A'_\mu = A'^a_\mu T_a$

gauge boson sector

generalized field strength tensor

$$F^a_{\mu\nu} T_a =: F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]$$

$$= \partial_\mu A^a_\nu T_a - \partial_\nu A^a_\mu T_a + ig \underbrace{[A^b_\mu T_b, A^c_\nu T_c]}_{i f_{bca} T_a A^b_\mu A^c_\nu}$$

$$= \underbrace{\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{abc} A_\mu^b A_\nu^c \right)}_{F_{\mu\nu}^a} T_a$$

$F_{\mu\nu}$ transforms homogeneously: $F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$ (exercise)

gauge boson term in the Lagrangean:

$$\mathrm{Tr}(T_a T_b) = \Lambda \delta_{ab}$$

$$\to \mathcal{L}_A = -\frac{1}{4\Lambda} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

$\to$ kinetic terms + <u>self couplings</u> of the gauge bosons