10. Feynman graphs

expressions in pert. expansion of S-matrix elements in (spinor) QED have the typical form

$$\underbrace{\langle 0 | a \ldots d \ldots b \ldots}_{\text{final state}} \; \underbrace{T\, \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x) \ldots}_{\text{interaction terms}} \; \underbrace{a^\dagger \ldots b^\dagger \ldots d^\dagger \ldots | 0 \rangle}_{\text{initial state}}$$

$\rightarrow$ vacuum expectation value of a sum of <u>products</u> of <u>creation</u>- and <u>annihilation-operators</u>

<u>Wick's theorem</u> allows to evaluate such expressions in a <u>systematic</u> way:

let $A_1, \ldots, A_n$ be a set of n creation- and annihilation operators $\Rightarrow$ $A_1 \ldots A_n = {:}A_1 \ldots A_n{:} +$

$+ \; {:}\underbrace{A_1 A_2}_{} A_3 \ldots A_n{:} + \ldots$ one contraction

$+ \; {:}\underbrace{A_1 A_2}_{} \underbrace{A_3 A_4}_{} \ldots A_n{:} + \ldots$ two contractions

$+ \ldots$ all possible contractions

where $\underbrace{A_1 A_2}_{} := \langle 0 | A_1 A_2 | 0 \rangle$

$$\underbracket{A_1 A_2 \cdots A_{m-1} A_m} A_{m+1} \cdots A_n =$$

$$= \pm \langle 0 | A_1 A_m | 0 \rangle \; A_2 \cdots A_{m-1} A_{m+1} \cdots A_n$$

the sign is $\pm$ depending on whether an even or odd number of <u>fermion</u> operators exchange position (no sign change for bosonic operators)

<u>proof</u> of Wick's theorem:

Wick's theorem is obviously true for <u>two</u> operators:

$$A_1 A_2 = :A_1 A_2: + \underbracket{A_1 A_2}$$

(a) for two bosonic operators, we have

$$:A_1 A_2: = \begin{cases} A_1 A_2 & \text{if } A_2 \text{ is annihilation op.} \\ A_2 A_1 & \text{if } A_2 \text{ is creation op.} \end{cases}$$

in the first case, Wick's theorem is fulfilled as $\langle 0 | A_1 A_2 | 0 \rangle = 0$

in the second case, we write

$$A_2 A_1 = A_1 A_2 + \underbrace{[A_2, A_1]}_{c\text{-number}} =$$

$$= A_1 A_2 + \langle 0 | [A_2, A_1] | 0 \rangle$$

$$= A_1 A_2 - \langle 0 | A_1 A_2 | 0 \rangle$$

(B) two fermionic operators:

$$:A_1 A_2: = \begin{cases} A_1 A_2 & \text{if } A_2 \text{ annihilation op.} \\ -A_2 A_1 & \text{if } A_2 \text{ creation op.} \end{cases}$$

in the latter case we write

$$- A_2 A_1 = A_1 A_2 - \underbrace{\{A_2, A_1\}}_{c\text{-number}}$$

$$= A_1 A_2 - \langle 0 | \{A_2, A_1\} | 0 \rangle$$

$$= A_1 A_2 - \langle 0 | A_1 A_2 | 0 \rangle$$

same as before

(c) if one operator is bosonic and the other fermionic, the formula is, of course, also valid

in the <u>next step</u>, we show the following:

$$:A_1 \dots A_n: C = :A_1 \dots A_n C: + :\underbracket{A_1 \dots A_n C}: + \dots + :A_1 \dots \underbracket{A_n C}:$$

<u>proof:</u> if C is an annihilation operator, then the formula is trivially true, as C is already in normal order and all contractions vanish

→ assume: C is a <u>creation</u> operator

→ commute (or anticommute) C to the left

$$:A_1 \dots A_n: C = \eta_{AB} \, B_1 \dots B_n C$$

$\{B_1 \dots B_n\}$ is $\{A_1 \dots A_n\}$ in normal order

$$B_n C = \eta_{nC}\, C B + [B_n, C]_{\mp}$$

$$= \eta_{nC}\, C B + \langle 0 | [B_n, C]_{\mp} | 0 \rangle$$

$$= \eta_{nC}\, C B + \langle 0 | B_n C | 0 \rangle$$

$\uparrow$

C is creation op.

$$\eta_{nC} = \begin{cases} -1 & \text{if } B_n, C \text{ fermionic} \\ & \text{otherwise} \end{cases}$$

$$= \eta_{nC}\, C B + \underbrace{B_n C}$$

$$\Rightarrow \; :A_1 \ldots A_n: C = \eta_{AB}\, \eta_{nC}\, B_1 \ldots C B_n +$$

$$+ \eta_{AB}\, B_1 \ldots \underbrace{B_n C}$$

continue...

$\downarrow$

$$= \eta_{AB}\, \eta_{nC} \ldots \eta_{1C}\; C B_1 \ldots B_n + \eta_{AB}\, B_1 \ldots \underbrace{B_n C}$$

$$+ \eta_{AB}\, B_1 \ldots \underbrace{B_{n-1} B_n C} + \ldots$$

$$= :A_1 \ldots A_n C: + :A_1 \ldots \underbrace{A_n C}: + \ldots +$$

$$+ :\underbrace{A_1 \ldots A_n C}:$$

final step: <u>induction</u>

Wick's theorem is valid for $n=2$; assuming that it is also valid for n, the step to $n+1$ uses the previous formula ✓

taking the <u>vacuum expectation value</u>, Wick's theorem simplifies to the following statement:

$$\langle 0 | A_1 \ldots A_n | 0 \rangle = \text{sum of all possible pairings}$$

example:

$$\langle 0 | A_1 A_2 A_3 A_4 | 0 \rangle = A_1 A_2 A_3 A_4 + A_1 A_2 A_3 A_4$$

$$+ A_1 A_2 A_3 A_4$$

<u>remark</u>: Wick's theorem also holds when any A_m is replaced by a <u>linear combination</u> of creation and annihilation operators

(spinor) QED:

$$\langle 0 | a \ldots d \ldots b \ldots T\, \bar{\psi}_a(x)\, A_\mu(x)\, \psi_b(x) \ldots b^\dagger \ldots d^\dagger \ldots a^\dagger \ldots | 0 \rangle$$

(in practice only <u>two</u> creation operators in <u>initial</u> state)

- <u>possible contractions:</u>

a) contraction <u>inside</u> the T-product

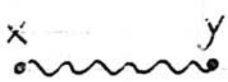
$\rightarrow$ <u>internal</u> lines (<u>virtual</u> particles)

$$\langle 0 | T A_\mu(x)\, A_\nu(y) | 0 \rangle = i D_{\mu\nu}(x-y) =$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{-i}{k^2 + i\varepsilon} \left[g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right]$$

- <u>photon propagator</u>

graphical representation:



space-time diagram

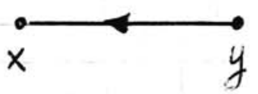
or

momentum-space diagram

$$\langle 0 | T \psi_a(x)\, \bar{\psi}_b(y) | 0 \rangle = -i S(x-y) =$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \frac{i}{\cancel{k} - m + i\varepsilon}$$

- fermion propagator



or

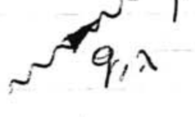
b) contractions of creation operators of the initial state with operators inside the T-product

$\longrightarrow$ external lines (real particles in the initial state)

$$\langle 0 | A_\mu(x)\, a(q,\lambda)^\dagger | 0 \rangle =$$

$$= \sum_{\lambda'} \int d\mu(k)\, e^{-ikx}\, \varepsilon_\mu(k,\lambda')\, \underbrace{\langle 0 | a(k,\lambda')\, a(q,\lambda)^\dagger | 0 \rangle}_{\delta(k,q)\, \delta_{\lambda'\lambda}}$$

$$= e^{-iqx}\, \varepsilon_\mu(q,\lambda)$$



incoming photon

$$\langle 0 | \psi_a(x) \, b(p,s)^\dagger | 0 \rangle =$$

$$= \sum_{s'} \int d\mu(k) \, e^{-ikx} u_a(k,s') \underbrace{\langle 0 | b(k,s') \, b(p,s)^\dagger | 0 \rangle}_{\delta(k,p)\, \delta_{s's}}$$

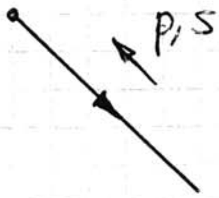
$$= e^{-ipx} \, u_a(p,s)$$

<u>incoming fermion</u>

$$\langle 0 | \overline{\psi}_a(x) \, d(p,s)^\dagger | 0 \rangle =$$

$$= \sum_{s'} \int d\mu(k) \, e^{-ikx} \, \overline{v}_a(k,s') \underbrace{\langle 0 | d(k,s') \, d(p,s)^\dagger | 0 \rangle}_{\delta(k,p)\, \delta_{s's}}$$

$$= e^{-ipx} \, \overline{v}_a(p,s)$$



<u>incoming anti-fermion</u>

c) contractions of annihilation operators of the final state with operators inside the T-product

$\longrightarrow$ <u>external</u> lines (<u>real</u> particles in the <u>final</u> state)

$$\langle 0 |\, a(q,\lambda)\; A_\mu(x)\, | 0 \rangle =$$

$$= \sum_{\lambda'} \int d\mu(k)\; e^{+ikx}\; \varepsilon_\mu(k,\lambda')^* \; \underbrace{\langle 0 |\, a(q,\lambda)\, a(k,\lambda')^\dagger | 0 \rangle}_{\delta(q,k)\,\delta_{\lambda\lambda'}}$$

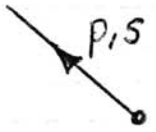
$$= e^{iqx}\; \varepsilon_\mu(q,\lambda)^*$$

final state photon

$$\langle 0 |\, b(p,s)\; \overline{\psi}_a(x)\, | 0 \rangle =$$

$$= \sum_{s'} \int d\mu(k)\; e^{+ikx}\; \overline{u}_a(k,s')\; \underbrace{\langle 0 |\, b(p,s)\; b(k,s')^\dagger | 0 \rangle}_{\delta(p,k)\,\delta_{ss'}}$$

$$= e^{+ipx}\; \overline{u}_a(p,s)$$

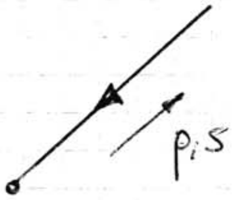


<u>fermion</u> in final state

$$\langle 0 | d(p,s)\, \psi_a(x) | 0 \rangle =$$

$$= \sum_{s'} \int d\mu(k)\, e^{+ikx}\, v_a(k,s') \underbrace{\langle 0 | d(p,s)\, d(k,s')^\dagger | 0 \rangle}_{\delta(p,k)\,\delta_{ss'}}$$

$$= e^{+ipx}\, v_a(p,s)$$



<u>anti</u>-<u>fermion</u> in final state

d) contractions of annihilation operators of the final state and creation operators of the initial state lead to disconnected diagrams $\rightarrow$ can be ignored

<u>remark</u>: Feynman rule for <u>interaction vertex</u> in (spinor) QED can be easily deduced from the (kinematically forbidden) process

$$f(p_1, s_1) + \gamma^*(k,\lambda) \rightarrow f(p_2, s_2)$$

$$\mathcal{L}_{int} = - j_\mu A^\mu = - q_f e \, \overline{\psi} \gamma_\mu \psi A^\mu$$

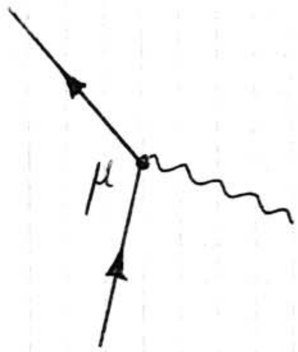
$$S = T e^{i \int d^4x \, \mathcal{L}_{int}}$$

$$\rightarrow \; - i q_f e \int d^4x \, \langle 0 | b(p_2, s_2) \, \overline{\psi}(x) \gamma_\mu \psi(x) A^\mu(x) \, b(p_1, s_1)^\dagger a(k, \lambda)^\dagger | 0 \rangle$$

$$= - i q_f e \int d^4x \, e^{-ikx} \varepsilon^\mu(k, \lambda) \; e^{+ip_2 x} \, \overline{u}(p_2, s_2) \gamma_\mu u(p_1, s_1) \, e^{-ip_1 x}$$

$$= \; (2\pi)^4 \delta^{(4)}(p_1 + k - p_2) \; \overline{u}(p_2, s_2) \, (-i q_f e \gamma_\mu) \, u(p_1, s_1)$$

$\uparrow$



$-i q_f e \gamma_\mu$

<u>remark:</u> $f + \gamma \rightarrow f$ kinematically forbidden as $p_1 + k = p_2$ (energy-momentum conservation) cannot be fulfilled, but factor $-i q_f e \gamma_\mu$, of course, unchanged in kinematically allowed processes (remember: $e^- e^+ \rightarrow \mu^- \mu^+$)

Feynman rules for the computation of $i\mathcal{M}_{fi}$
$\left(S_{fi} = \delta_{fi} + i\,(2\pi)^4\,\delta^{(4)}(P_i - P_f)\,\mathcal{M}_{fi} \right)$ in
(spinor) QED

$u(p,s)$ fermion with momentum p and spin pol. s in the initial state

$\bar{v}(p,s)$ anti-fermion with momentum p and spin pol. s in the initial state

$\bar{u}(p,s)$ fermion with momentum p and spin pol. s in the final state

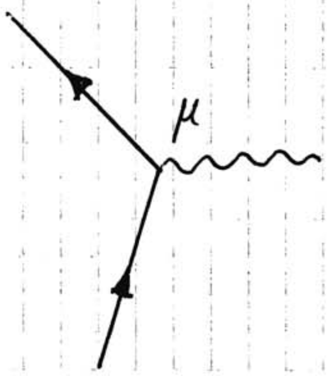
$v(p,s)$ anti-fermion with momentum p and spin pol. s in the final state

$\varepsilon_\mu(k,\lambda)$ photon with momentum k and pol. λ in the initial state

$\varepsilon_\mu(k,\lambda)^*$ photon with momentum k and pol. λ in the final state

$$\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$$ internal photon line

$$\frac{i}{\not{p}-m+i\varepsilon}$$ internal fermion line



$-iqe\gamma_\mu$ interaction vertex