

9. Production of fermion pairs in electron-positron collisions

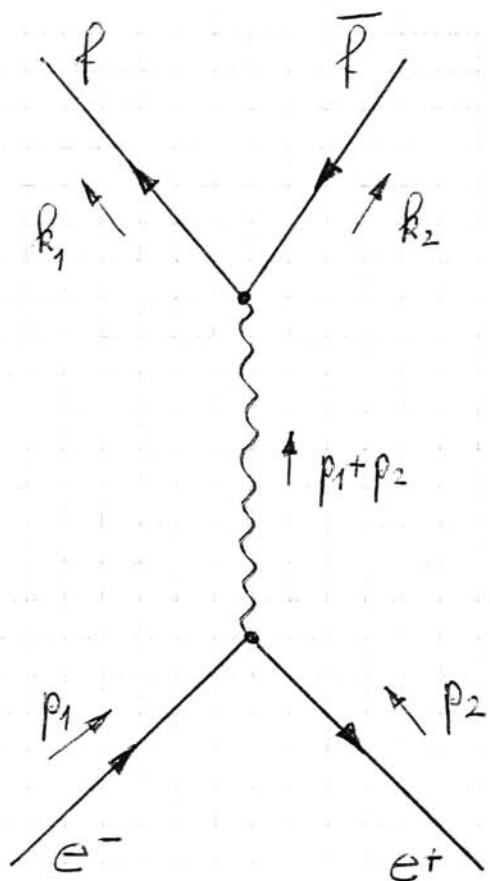
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the result of the previous section may, of course, be applied to the pair production $e^-e^+ \rightarrow f\bar{f}$ ($f \neq e$) of arbitrary (charged) spin $1/2$ fermions (except $f=e$)

e.g. $f = \mu, \tau, u, d, s, \dots$

only modification: $ie\gamma_\nu \rightarrow -ieq_f\gamma_\nu$ where q_f is the electric charge of the fermion field f in units of the elementary charge e (e.g. $q_u = 2/3, q_d = -1/3$)

→ invariant amplitude for $e^-(p_1)e^+(p_2) \rightarrow f(k_1)\bar{f}(k_2)$



$$\begin{aligned} iM &= \bar{v}(p_2, \lambda_2; m_e) ie\gamma_\mu u(p_1, \lambda_1; m_e) \\ &\times \frac{-ieq_f\gamma_\nu}{(p_1 + p_2)^2} \\ &\times \bar{u}(k_1, \lambda_1; m_f) (-ieq_f\gamma_\nu) v(k_2, \lambda_2; m_f) \end{aligned}$$

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consider the case of unpolarized e^+e^- beams

→ mixed state $\downarrow\downarrow, \downarrow\uparrow, \uparrow\downarrow, \uparrow\uparrow$

→ we have to average over the spin polarizations in the initial state:

$$\frac{1}{4} \sum_{s_1, s_2} |M|^2$$

if the detector is unable to distinguish the spins of the final state particles, we have to sum over the spin polarizations in the final state:

$$\sum_{r_1, r_2} \dots$$

$$\Rightarrow \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4 g_F^2}{4 [(p_1 + p_2)^2]^2} \sum_{\text{spins}}$$

$$\bar{v}(p_2, s_2; m_e) \gamma^\mu u(p_1, s_1; m_e) u^\dagger(p_1, s_1; m_e) (g^\mu)^\nu (\gamma^\nu)^\dagger \gamma^\nu v(p_2, s_2; m_e) \\ \bar{u}(k_1, r_1; m_f) \gamma_\mu v(k_2, r_2; m_f) \bar{v}(k_2, r_2; m_f) \gamma_\nu u(k_1, r_1; m_f)$$

$$= \frac{e^4 q_f^2}{4 (p_1 + p_2)^4} \sum_{\text{spins}}$$

$$\text{Tr} [v(p_2, s_2; m_e) \bar{v}(p_2, s_2; m_e) \gamma^\mu u(p_1, s_1; m_e) \bar{u}(p_1, s_1; m_e) \gamma^\nu]$$

$$\text{Tr} [u(k_1, r_1; m_f) \bar{u}(k_1, r_1; m_f) \gamma_\mu v(k_2, r_2; m_f) \bar{v}(k_2, r_2; m_f) \gamma_\nu]$$

$$= \frac{e^4 q_f^2}{4 (p_1 + p_2)^4} \text{Tr} [(p_2 - m_e) \gamma^\mu (p_1 + m_e) \gamma^\nu] \times$$

$$\times \text{Tr} [(k_1 + m_f) \gamma_\mu (k_2 - m_f) \gamma_\nu]$$

$$= \frac{e^4 q_f^2}{4 (p_1 + p_2)^4} \text{Tr} (p_2 \gamma^\mu p_1 \gamma^\nu - m_e^2 \gamma^\mu \gamma^\nu) \times$$

$$\times \text{Tr} (k_1 \gamma_\mu k_2 \gamma_\nu - m_f^2 \gamma_\mu \gamma_\nu)$$

traces of γ matrices

1) trace of a product of an odd number of γ matrices vanishes

$$2) \text{Tr} (\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

$$3) \text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$

$$\Rightarrow \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{4e^4 q_f^2}{(p_1 + p_2)^4} \times$$

$$\times [p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - p_1 \cdot p_2 g^{\mu\nu} - m_e^2 g^{\mu\nu}] \times$$

$$\times [k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - k_1 \cdot k_2 g_{\mu\nu} - m_f^2 g_{\mu\nu}]$$

$$= \frac{4e^4 q_f^2}{(p_1 + p_2)^4} \left\{ k_1 \cdot p_2 p_1 \cdot k_2 + p_2 \cdot k_2 p_1 \cdot k_1 \right.$$

$$- (k_1 \cdot k_2 + m_f^2) p_1 \cdot p_2 + p_1 \cdot k_1 p_2 \cdot k_2 + p_1 \cdot k_2 k_1 \cdot p_2$$

$$- (k_1 \cdot k_2 + m_f^2) p_1 \cdot p_2 - (p_1 \cdot p_2 + m_e^2) [2 k_1 \cdot k_2$$

$$- 4 (k_1 \cdot k_2 + m_f^2)] \left. \right\}$$

$$= \frac{4e^4 q_f^2}{(p_1 + p_2)^4} \left\{ 2 p_1 \cdot k_1 p_2 \cdot k_2 + 2 p_1 \cdot k_2 k_1 \cdot p_2 \right.$$

$$+ 2 m_f^2 p_1 \cdot p_2 + 2 m_e^2 k_1 \cdot k_2 + 4 m_e^2 m_f^2 \left. \right\}$$

$$= \frac{8e^4 q_f^2}{(p_1 + p_2)^4} \left\{ p_1 \cdot k_1 p_2 \cdot k_2 + p_1 \cdot k_2 k_1 \cdot p_2 \right.$$

$$+ m_f^2 p_1 \cdot p_2 + m_e^2 k_1 \cdot k_2 + 2 m_e^2 m_f^2 \left. \right\}$$

Kinematics of $e^-(p_1) e^+(p_2) \rightarrow f(k_1) \bar{f}(k_2)$

energy - momentum conservation:

$$p_1 + p_2 = k_1 + k_2$$

invariants:

$$s = (p_1 + p_2)^2 = 2m_e^2 + 2p_1 \cdot p_2$$

$$= (k_1 + k_2)^2 = 2m_f^2 + 2k_1 \cdot k_2$$

$$t = (p_1 - k_1)^2 = m_e^2 + m_f^2 - 2p_1 \cdot k_1$$

$$= (k_2 - p_2)^2 = m_e^2 + m_f^2 - 2p_2 \cdot k_2$$

the other invariants can be expressed through s, t :

$$p_1 + p_2 - k_1 = k_2$$

$$2m_e^2 + m_f^2 + 2p_1 \cdot p_2 - 2p_1 \cdot k_1 - 2p_2 \cdot k_1 = m_f^2$$

$$\Rightarrow s + t - m_e^2 - m_f^2 - 2p_2 \cdot k_1 = 0$$

$$\Rightarrow 2p_2 \cdot k_1 = s + t - m_e^2 - m_f^2$$

analogously:

$$p_1 + p_2 - k_2 = k_1$$

$$\Rightarrow 2 p_1 \cdot k_2 = s + t - m_e^2 - m_f^2$$

$$\Rightarrow \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4 g_f^2}{s^2} \left\{ \frac{1}{4} (m_e^2 + m_f^2 - t)^2 \right.$$

$$+ \frac{1}{4} (s + t - m_e^2 - m_f^2)^2 + m_f^2 \frac{1}{2} (s - 2m_e^2)$$

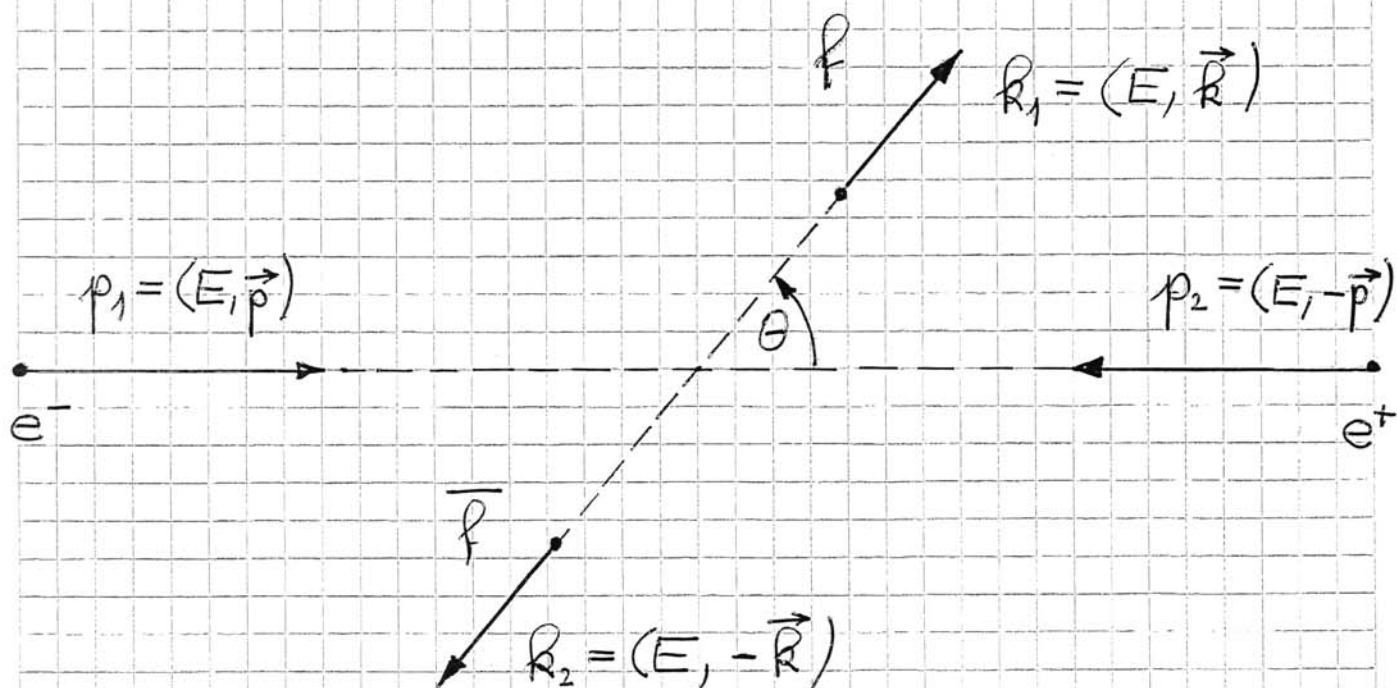
$$+ m_e^2 \frac{1}{2} (s - 2m_f^2) + 2 m_e^2 m_f^2 \left. \right\}$$

$$= \frac{2e^4 g_f^2}{s^2} \left\{ (m_e^2 + m_f^2 - t)^2 + (s + t - m_e^2 - m_f^2)^2 \right. \\ \left. + 2s (m_e^2 + m_f^2) \right\}$$

$$= \frac{2e^4 g_f^2}{s^2} \left\{ 2 (m_e^2 + m_f^2 - t)^2 + s^2 + 2s (t - m_e^2 - m_f^2) \right. \\ \left. + 2s (m_e^2 + m_f^2) \right\}$$

$$= \frac{2e^4 g_f^2}{s^2} \left\{ s^2 + 2st + 2 (m_e^2 + m_f^2 - t)^2 \right\}$$

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CMS

$$s = (p_1 + p_2)^2 = (2E)^2 \quad 2E = \sqrt{s}$$

$$t = (p_1 - k_1)^2 = m_e^2 + m_f^2 - 2(E^2 - \vec{p} \cdot \vec{k})$$

$$= m_e^2 + m_f^2 - \frac{s}{2} + 2|\vec{p}| |\vec{k}| \cos \theta$$

$$\Rightarrow \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{2e^4 q_f^2}{s^2} \left\{ \cancel{s^2} + 2s \left(m_e^2 + m_f^2 - \cancel{\frac{s}{2}} + \cancel{2|\vec{p}| |\vec{k}| \cos \theta} \right) \right.$$

$$\left. + \frac{s^2}{2} + 8|\vec{p}|^2 |\vec{k}|^2 \cos^2 \theta - 4s|\vec{p}| |\vec{k}| \cos \theta \right\}$$

$$= \frac{2e^4 q_f^2}{s^2} \left\{ \frac{s^2}{2} + 2s (m_e^2 + m_f^2) + \underbrace{8|\vec{p}|^2}_{E^2 - m_e^2} \underbrace{|\vec{k}|^2}_{E^2 - m_f^2} \cos^2 \theta \right\}$$

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cross section

$$d\sigma = \frac{1}{4 \sqrt{(p_1 \cdot p_2)^2 - m_e^4}} \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \frac{d^3 k_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \times \frac{1}{4} \sum_{\text{spins}} |M|^2$$

$$(p_1 \cdot p_2)^2 - m_e^4 = \left(\frac{s}{2} - m_e^2\right)^2 - m_e^4 = \frac{s^2}{4} - s m_e^2$$

$$\Rightarrow 4 \sqrt{(p_1 \cdot p_2)^2 - m_e^4} = 2 \sqrt{s(s - 4m_e^2)}$$

$$\int \frac{d^3 k_1}{2k_1^0} \frac{d^3 k_2}{2k_2^0} \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \dots$$

$$= \int \frac{d^3 k_1}{2k_1^0} d^4 k_2 \Theta(k_2^0) \delta(k_2^2 - m_e^2) \delta(p_1 + p_2 - k_1 - k_2) \dots$$

$$= \int \frac{d^3 k_1}{2k_1^0} \Theta(p_1^0 + p_2^0 - k_1^0) \delta[(p_1 + p_2 - k_1)^2 - m_e^2] \dots$$

$$\stackrel{\text{CMS}}{\downarrow} = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\theta \int_0^{\infty} \frac{d|\vec{k}_1| |\vec{k}_1|^2}{2k_1^0} \Theta(\sqrt{s} - k_1^0) \times$$

$$\times \delta(s - 2\sqrt{s} k_1^0) \dots$$

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$$= \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\Theta \int_{m_f}^{\infty} \frac{dK_1^0 K_1^0 |\vec{K}_1|}{2 K_1^0} \Theta(\sqrt{s} - K_1^0) \times$$

$$\times \frac{1}{2\sqrt{s}} \delta(K_1^0 - \frac{\sqrt{s}}{2}) \dots$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\Theta \frac{\sqrt{\frac{s}{4} - m_f^2}}{4\sqrt{s}} \underbrace{\Theta(\frac{\sqrt{s}}{2})}_{1} \Theta(\frac{\sqrt{s}}{2} - m_f) \dots$$

$$\Rightarrow \frac{d\sigma}{d\cos\Theta} = \frac{1}{2\sqrt{s(s-4m_e^2)}} \frac{1}{(2\pi)^2} \underset{\substack{\text{integration over } \varphi \\ \downarrow}}{2\pi} \frac{\sqrt{\frac{s}{4} - m_f^2}}{4\sqrt{s}}$$

$$\Theta(\frac{\sqrt{s}}{2} - m_f) \frac{2e^4 q_f^2}{s^2} \left\{ \frac{s^2}{2} + 2s(m_e^2 + m_f^2) \right.$$

$$\left. + 8 \underbrace{\left(\frac{s}{4} - m_e^2\right)}_{|\vec{p}|^2} \underbrace{\left(\frac{s}{4} - m_f^2\right)}_{|\vec{K}|^2} \cos^2\Theta \right\}$$

$$e^2 = 4\pi\alpha$$

$$\frac{d\sigma}{d\cos\Theta} = \frac{2\alpha^2 q_f^2 \pi \Theta\left(\frac{\sqrt{s}}{2} - m_f\right) \sqrt{\frac{s}{4} - m_f^2}}{s^3 \sqrt{s - 4m_e^2}} \times \left\{ \frac{s}{2} + 2s(m_e^2 + m_f^2) + 8\left(\frac{s}{4} - m_e^2\right)\left(\frac{s}{4} - m_f^2\right)\cos^2\Theta \right\}$$

special case $m_e \ll m_f < \sqrt{s}/2$ $\rightarrow$ neglect m_e

$$\rightarrow \frac{d\sigma}{d\cos\Theta} \simeq \frac{2\alpha^2 q_f^2 \pi \Theta\left(\frac{\sqrt{s}}{2} - m_f\right) \sqrt{\frac{s}{4} - m_f^2}}{s^2 \sqrt{s}}$$

$$\left\{ \frac{s}{2} + 2m_f^2 + 2\left(\frac{s}{4} - m_f^2\right)\cos^2\Theta \right\}$$

$$\sigma_{\text{tot}} = \frac{4\alpha^2 q_f^2 \pi \sqrt{\frac{s}{4} - m_f^2}}{s^2 \sqrt{s}} \left\{ \frac{s}{2} + 2m_f^2 + \frac{2}{3}\left(\frac{s}{4} - m_f^2\right) \right\}$$

high energy behaviour ($\sqrt{s} \gg 2m_f$)

$$\frac{d\sigma}{d\cos\Theta} \rightarrow \frac{\alpha^2 q_f^2 \pi}{2s} (1 + \cos^2\Theta)$$

$$\sigma_{\text{tot}} \rightarrow \frac{4\alpha^2 q_f^2 \pi}{3s}$$

$$d\sigma/d\Omega \rightarrow \frac{\alpha^2 q_f^2}{4s} (1 + \cos^2\Theta)$$

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$e^+ e^- \rightarrow \text{hadrons}$

$$e^+ e^- \rightarrow q_i \bar{q}_i \quad \begin{array}{l} i = 1, 2, 3 \\ \text{colour index} \end{array}$$

$u \ c \ t \quad q_f = 2/3$

$d \ s \ b \quad q_f = -1/3$

$$R := \frac{\sigma(e^+ e^- \rightarrow \text{hadrons})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)}$$

naive expectation (strong interaction switched off)

$$R \xrightarrow[\substack{\frac{\sqrt{s}}{2} \gg m_\mu, m_q}]{\substack{\uparrow \\ \text{colour}}} 3 \sum_{q_f} q_f^2$$

below $c\bar{c}$ threshold:

$$R \approx 3 \left[\underset{\uparrow}{\left(\frac{2}{3}\right)^2} + \underset{\uparrow}{\left(-\frac{1}{3}\right)^2} + \underset{\uparrow}{\left(-\frac{1}{3}\right)^2} \right] = 2$$

$u \qquad d \qquad s$

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above $c\bar{c}$ threshold, but below $b\bar{b}$ threshold:

$$R \approx 3 \left[\underbrace{2 \times \left(\frac{2}{3}\right)^2}_{u, c} + \underbrace{2 \times \left(-\frac{1}{3}\right)^2}_{d, s} \right] = \frac{10}{3}$$

above $b\bar{b}$ threshold:

$$R \approx 3 \left[\underbrace{2 \times \left(\frac{2}{3}\right)^2}_{u, c} + \underbrace{3 \times \left(-\frac{1}{3}\right)^2}_{d, s, b} \right] = \frac{11}{3}$$