8. Quantum electrodynamis (QED) of spin 1/2 fermions

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - j_\mu A^\mu$$

$j_\mu = q\,\bar{\psi}\gamma_\mu\psi$ electromagnetic current

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ electromagnetic field strength tensor

in the case of several types of charged spin 1/2 fields:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_f \bar{\psi}_f (i\not{\partial} - m_f)\psi_f - j_\mu A^\mu$$

$$j_\mu = \sum_f q_f \bar{\psi}_f \gamma_\mu \psi_f$$

(e.g. $f = e, \mu, \ldots$)

$\mathcal{L}$ is invariant under <u>local</u> $U(1)$ gauge transformations: $\psi_f \to e^{-i\alpha q_f}\psi_f$, $A_\mu \to A_\mu + \partial_\mu \alpha$

$$\bar{\psi} i \not{\partial} \psi \to \bar{\psi} i \not{\partial} \psi + q \bar{\psi} (\not{\partial} \alpha) \psi$$

$$-q \bar{\psi} \not{A} \psi \to -q \bar{\psi} \not{A} \psi - q \bar{\psi} (\not{\partial} \alpha) \psi$$

we define the <u>covariant derivative</u>

$$D_\mu = \partial_\mu + i q A_\mu$$

$$\bar{\psi} i \not{\partial} \psi - q \bar{\psi} \not{A} \psi = \bar{\psi} i \not{D} \psi$$

transform homogeneously under local gauge transformations

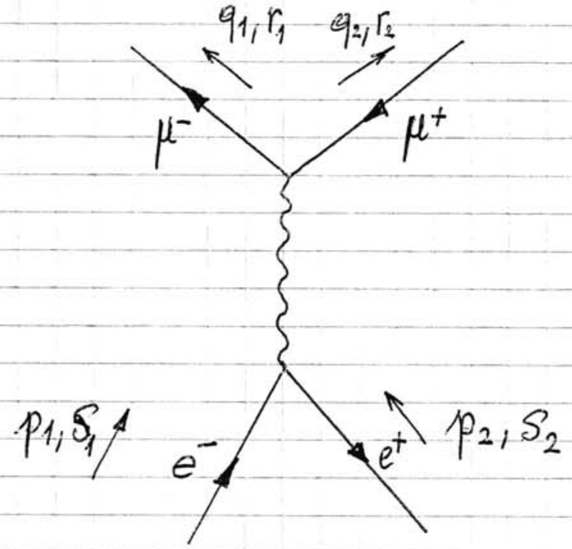
perturbative treatment of QED:

add gauge-breaking term

$$\mathcal{L} \to \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\xi}{2} (\partial_\mu A^\mu)^2 + \sum_f \bar{\psi}_f (i \not{\partial} - m_f) \psi_f}_{\mathcal{L}_0} \underbrace{- j_\mu A^\mu}_{\mathcal{L}_{int}}$$

$$S = T e^{-i \int d^4x \, j_\mu^{IP} A^\mu_{IP}}$$

$e^- e^+ \to \mu^- \mu^+$ (lowest order)



$$\langle \mu^- \mu^+ \text{ out} \mid e^- e^+ \text{ in} \rangle =$$

$$= \langle 0 | d^{(\mu)}(q_2, r_2)\, b^{(\mu)}(q_1, r_1)\, T e^{-i\int d^4x \left(j_\mu^{(e)}(x) + j_\mu^{(\mu)}(x)\right) A^\mu(x)}\, b(p_1, s_1)^\dagger d(p_2, s_2)^\dagger |0\rangle$$

$$= \frac{(-i)^2}{2!} \int d^4x\, d^4y\, \langle 0 | d^{(\mu)}(q_2, r_2)\, b^{(\mu)}(q_1, r_1)$$

$$T\left(j_\mu^{(e)}(x)\, j_\nu^{(\mu)}(y) + j_\mu^{(\mu)}(x)\, j_\nu^{(e)}(y) \right) A^\mu(x) A^\nu(y)$$

$$b^{(e)}(p_1, s_1)^\dagger\, d^{(e)}(p_2, s_2)^\dagger |0\rangle + \ldots$$

$$= (-i)^2 \int d^4x\, d^4y\, \langle 0 | d^{(\mu)}(q_2, r_2)\, b^{(\mu)}(q_1, r_1)\, j_\nu^{(\mu)}(y) | 0 \rangle$$

$$\langle 0 | j_\mu^{(e)}(x)\, b^{(e)}(p_1, s_1)^\dagger\, d^{(e)}(p_2, s_2)^\dagger |0\rangle \langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle$$

$+ \ldots$

$$\langle 0 | j_\mu(x)\, b(p_1, s_1)^\dagger\, d(p_2, s_2)^\dagger | 0 \rangle =$$

$$= -e \langle 0 | \overline{\psi}(x) \gamma_\mu \psi(x)\, b(p_1, s_1)^\dagger\, d(p_2, s_2)^\dagger | 0 \rangle$$

$$\cancel{b^\dagger} + d \qquad b + \cancel{d^\dagger}$$

$$= -e \sum_{s, s'} \int d\mu(k)\, d\mu(k')\, e^{-ik'x}\, e^{-ikx}$$

$$\overline{v}(k', s') \gamma_\mu u(k, s) \langle 0 | d(k', s') \underbrace{b(k, s)\, b(p_1, s_1)^\dagger}_{\substack{-b(p_1, s_1)^\dagger b(k, s) + \\ + \delta(k, p_1) \delta_{s s_1}}} d(p_2, s_2)^\dagger | 0 \rangle$$

$$= -e \sum_{s, s'} \int d\mu(k)\, d\mu(k')\, e^{-ik'x}\, e^{-ikx}\, \overline{v}(k', s') \gamma_\mu u(k, s)$$

$$\delta(k, p_1) \delta_{s s_1}\, \delta(k', p_2) \delta_{s' s_2}$$

$$= -e\, e^{-ip_2 x}\, e^{-ip_1 x}\, \overline{v}(p_2, s_2) \gamma_\mu u(p_1, s_1)$$

$$= e^{-ip_2 x}\, e^{-ip_1 x}\, \overline{v}(p_2, s_2; m_e) (-e \gamma_\mu) u(p_1, s_1; m_e)$$

analogously:

$$\langle 0 | d(q_2, r_2)\, b(q_1, r_1)\, j_\nu(y) | 0 \rangle$$

$$= -e \langle 0 | d(q_2, r_2)\, b(q_1, r_1)\, \underset{b^\dagger + \cancel{d}}{\overline{\psi}(y)}\, \gamma_\nu\, \underset{\cancel{b} + d^\dagger}{\psi(y)} | 0 \rangle$$

$$= -e \sum_{s,s'} \int d\mu(k)\, d\mu(k')\, e^{iky}\, e^{ik'y}\, \overline{u}(k', s')\, \gamma_\nu\, v(k, s)$$

$$\underbrace{\langle 0 | d(q_2, r_2)\, b(q_1, r_1)\, b(k', s')^\dagger\, d(k, s)^\dagger | 0 \rangle}_{\delta_{r_1 s'}\, \delta(q_1, k')\, \delta_{r_2 s}\, \delta(q_2, k)}$$

$$= e^{iq_1 y}\, e^{iq_2 y}\, \overline{u}(q_1, r_1; m_\mu)(-e\gamma_\nu)\, v(q_2, r_2; m_\mu)$$

$$\Rightarrow \langle \mu^-(q_1, r_1)\, \mu^+(q_2, r_2)\ \text{out} | e^-(p_1, s_1)\, e^+(p_2, s_2)\ \text{in} \rangle =$$

$$= \int d^4x\, d^4y\, e^{-i(p_1+p_2)x}\, e^{+i(q_1+q_2)y}\, i D^{\mu\nu}(x-y)$$

$$\overline{v}(p_2, s_2; m_e)\, (ie\gamma_\mu)\, u(p_1, s_1; m_e)$$

$$\overline{u}(q_1, r_1; m_\mu)\, (ie\gamma_\nu)\, v(q_2, r_2; m_\mu)$$

$$= \int d^4x\, d^4y\, \frac{d^4k}{(2\pi)^4}\, e^{-i(p_1+p_2)x}\, e^{+i(q_1+q_2)y}$$

$$\frac{-i\, e^{-ik(x-y)}}{k^2} \left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right) k^\mu k^\nu \right]$$

$$\bar{v}(p_2, s_2; m_e)\, (ie\gamma_\mu)\, u(p_1, s_1; m_e)$$

$$\bar{u}(q_1, r_1; m_\mu)\, (ie\gamma_\nu)\, v(q_2, r_2; m_\mu)$$

$$= (2\pi)^4\, \delta^{(4)}(p_1+p_2-q_1-q_2)$$

$$\frac{-i}{(p_1+p_2)^2} \left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right) (p_1+p_2)^\mu (p_1+p_2)^\nu \right]$$

$$\bar{v}(p_2, s_2; m_e)\, (ie\gamma_\mu)\, u(p_1, s_1; m_e)$$

$$\bar{u}(q_1, r_1; m_\mu)\, (ie\gamma_\nu)\, v(q_2, r_2; m_\mu)$$

gauge-dependent terms $\sim (p_1+p_2)^\mu (p_1+p_2)^\nu$

do <u>not</u> contribute (current conservation):

$$\bar{v}(p_2, s_2; m_e)\, (\underset{\downarrow \atop m_e}{\not{p}_1} + \underset{\downarrow \atop -m_e}{\not{p}_2})\, u(p_1, s_1; m_e) = 0$$

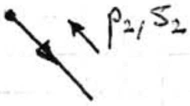
<u>observations</u> (Feynman rules in momentum space):

factor $(2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - q_1 - q_2)$ expresses

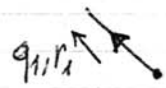
<u>energy-momentum conservation</u>

$u(p_1, s_1; m_e)$ <u>electron</u> with momentum p_1 and spin s_1 <u>in initial state</u>

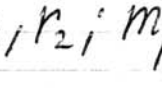
$\bar{v}(p_2, s_2; m_e)$ <u>positron</u> with momentum p_2 and spin s_2 <u>in initial state</u>



$\bar{u}(q_1, r_1; m_\mu)$ <u>μ^-</u> with momentum q_1 and spin r_1 <u>in final state</u>



$v(q_2, r_2; m_\mu)$ <u>μ^+</u> with momentum q_2 and spin r_2 <u>in final state</u>



$ie\gamma_\mu$ <u>interaction vertex</u>

$\dfrac{-i g_{\mu\nu}}{k^2}$ <u>photon propagator</u>

<u>internal</u> photon line

S-matrix element has the form

$$(2\pi)^4 \, \delta^{(4)}(p_1+p_2-q_1-q_2) \; i\mathcal{M}(p_1,p_2 \to q_1,q_2)$$

the relevant information about the scattering process is contained in the <u>invariant matrix-element (amplitude)</u> $\mathcal{M}(p_1,p_2 \to q_1,q_2)$

$\mathcal{M}$ is related to the (differential) <u>cross section</u>

$$d\sigma = \frac{S}{4\sqrt{(p_1\cdot p_2)^2 - p_1^2 p_2^2}} \, d\mu(q_1)\, d\mu(q_2)\, (2\pi)^4 \delta^{(4)}(p_1+p_2-q_1-q_2) \times |\mathcal{M}(p_1,p_2\to q_1,q_2)|^2$$

(we will show this later)

statistical factor $S = \begin{cases} 1 & \text{for } \underline{\text{distinguishable}} \text{ particles in final state} \\ \frac{1}{2!} & \text{for } \underline{\text{identical}} \text{ particles in final state} \end{cases}$

in our case $(e^-e^+ \to \mu^-\mu^+)$: $S = 1$