7. Perturbation theory

$H = H_0 + H_{int}$

H_0 ... Hamiltonian of free theory (time evolution known explicitly)

H_{int} ... interaction term

interaction picture

notation: IP ... interaction picture

S ... Schrödinger picture

H ... Heisenberg picture

time evolution of operators in the interaction picture defined by

$$A_{IP}(t) := e^{iH_0 t} A_S e^{-iH_0 t}$$

→ time evolution of state vectors in IP determined as time evolution of expectation values identical in all pictures

$$\langle \psi_S(0)| e^{iHt} A_S \underbrace{e^{-iHt} \overbrace{|\psi_S(0)\rangle}^{|\psi_H\rangle}}_{|\psi_S(t)\rangle}$$

$$= \langle \psi_S(0)| e^{iHt} e^{-iH_0 t} \underbrace{e^{iH_0 t} A_S e^{-iH_0 t}}_{A_{IP}(t)} \underbrace{e^{iH_0 t} e^{-iHt} |\psi_S(0)\rangle}_{|\psi_{IP}(t)\rangle}$$

$$|\psi_{IP}(t)\rangle = e^{iH_0 t} \underbrace{e^{-iHt} |\psi_S(0)\rangle}_{|\psi_S(t)\rangle} = \underbrace{e^{iH_0 t} e^{-iHt}}_{= U(t)} |\psi_H\rangle$$

differential equation for $U(t)$:

$$\dot{U}(t) = \frac{d}{dt}\left(e^{iH_0 t} e^{-iHt}\right) = e^{iH_0 t}(iH_0 - iH) e^{-iHt}$$

$$= -i\, e^{iH_0 t} H_{int} e^{-iHt} = -i \underbrace{e^{iH_0 t} H_{int} e^{-iH_0 t}}_{H_{int}^{IP}(t)} \underbrace{e^{iH_0 t} e^{-iHt}}_{U(t)}$$

$$\Leftrightarrow \quad i \frac{d}{dt} |\psi_{IP}(t)\rangle = H_{int}^{IP}(t)\, |\psi_{IP}(t)\rangle$$

$\uparrow$

time-dependent

relation between $|\psi_{IP}(t_0)\rangle$ and $|\psi_{IP}(t)\rangle$:

$$|\psi_{IP}(t)\rangle = e^{iH_0 t}\, e^{-iHt}\, |\psi_{IP}(0)\rangle = U(t)|\psi_{IP}(0)\rangle$$

$$|\psi_{IP}(t_0)\rangle = e^{iH_0 t_0}\, e^{-iHt_0}\, |\psi_{IP}(0)\rangle = U(t_0)|\psi_{IP}(0)\rangle$$

$$\Rightarrow \quad |\psi_{IP}(t)\rangle = U(t)\, U(t_0)^{-1}\, U(t_0)\, |\psi_{IP}(0)\rangle$$

$$= \underbrace{U(t)\, U(t_0)^{-1}}_{=:\, U(t,t_0)}\, |\psi_{IP}(t_0)\rangle$$

$$U(t,t_0) = e^{iH_0 t}\, e^{-iHt}\, e^{+iHt_0}\, e^{-iH_0 t_0}$$

$$\frac{\partial U(t,t_0)}{\partial t} = -i\, H_{int}^{IP}(t)\, U(t,t_0)$$

initial condition $U(t_0,t_0) = \mathbb{1}$

$$\int_{t_0}^{t} d\tau \;\Big|\; \frac{\partial U(\tau,t_0)}{\partial \tau} = -i\, H_{int}^{IP}(\tau)\, U(\tau,t_0)$$

$$U(t,t_0) - \underbrace{U(t_0,t_0)}_{\mathbb{1}} = -i \int_{t_0}^{t} d\tau\, H_{int}(\tau)\, U(\tau,t_0)$$

this integral equation can be solved by

<u>iteration</u>:

$$U(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} d\tau \; H_{int}^{IP}(\tau)\, U(\tau,t_0)$$

$$= \mathbb{1} - i\int_{t_0}^{t} d\tau \; H_{int}^{IP}(\tau)$$

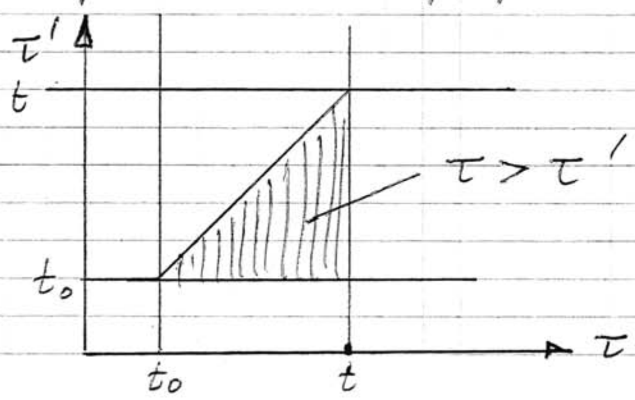
$$+ (-i)^2 \int_{t_0}^{t} d\tau \; H_{int}^{IP}(\tau) \int_{t_0}^{\tau} d\tau' \, H_{int}^{IP}(\tau')\, U(\tau',t_0)$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} d\tau_n \int_{t_0}^{\tau_n} d\tau_{n-1} \ldots \int_{t_0}^{\tau_2} d\tau_1 \; H_{int}^{IP}(\tau_n) \ldots H_{int}^{IP}(\tau_1)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} d\tau_n \ldots \int_{t_0}^{t} d\tau_1 \; T\left(H_{int}^{IP}(\tau_n) \ldots H_{int}^{IP}(\tau_1)\right)$$

$$=: T\, e^{-i\int_{t_0}^{t} d\tau \, H_{int}^{IP}(\tau)}$$

explanation of the last step for $n=2$:



$$\int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau' \; H_{int}^{IP}(\tau) \, H_{int}^{IP}(\tau') \qquad \tau > \tau'$$

$$= \int_{t_0}^{t} d\tau' \int_{t_0}^{\tau'} d\tau \; H_{int}^{IP}(\tau') \, H_{int}^{IP}(\tau) \qquad \tau' > \tau$$

$$= \frac{1}{2} \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau' \; T\left(H_{int}^{IP}(\tau) \, H_{int}^{IP}(\tau') \right)$$

remarks: a) the time evolution operator in the IP has the property $U(t,t_0) = U(t,t_1)\, U(t_1,t_0)$

b) the interaction picture may also be used in the case of a perturbation with an explicit time-dependence

$$H_S = H_0 + H_1(t)$$

time-evolution in IP: $U(t,t_0) = T \, e^{-i\int_{t_0}^{t} d\tau \, H_1^{IP}(\tau)}$

where $H_1^{IP}(t) = e^{iH_0 t} H_1(t) \, e^{-iH_0 t}$

for $H_1(t) \xrightarrow[t\to\pm\infty]{} 0$, the <u>vacuum-to-vacuum transition amplitude</u> is given by

$$\langle 0 | T e^{-i\int_{-\infty}^{+\infty} dt\, H_1^{IP}(t)} | 0 \rangle$$

where $H_0 |0\rangle = 0$ ($|0\rangle$ denotes the ground state of the Hamiltonian H_0)

example: $\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2) - \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_0} + \underbrace{f(x)\phi}_{\mathcal{L}_1}$

$$H_0 = \int d^3x \left\{ \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right) + \frac{\lambda}{4!}\phi^4 \right\}$$

$$H_1(t) = -\int d^3x\, f(t,\vec{x})\,\phi$$

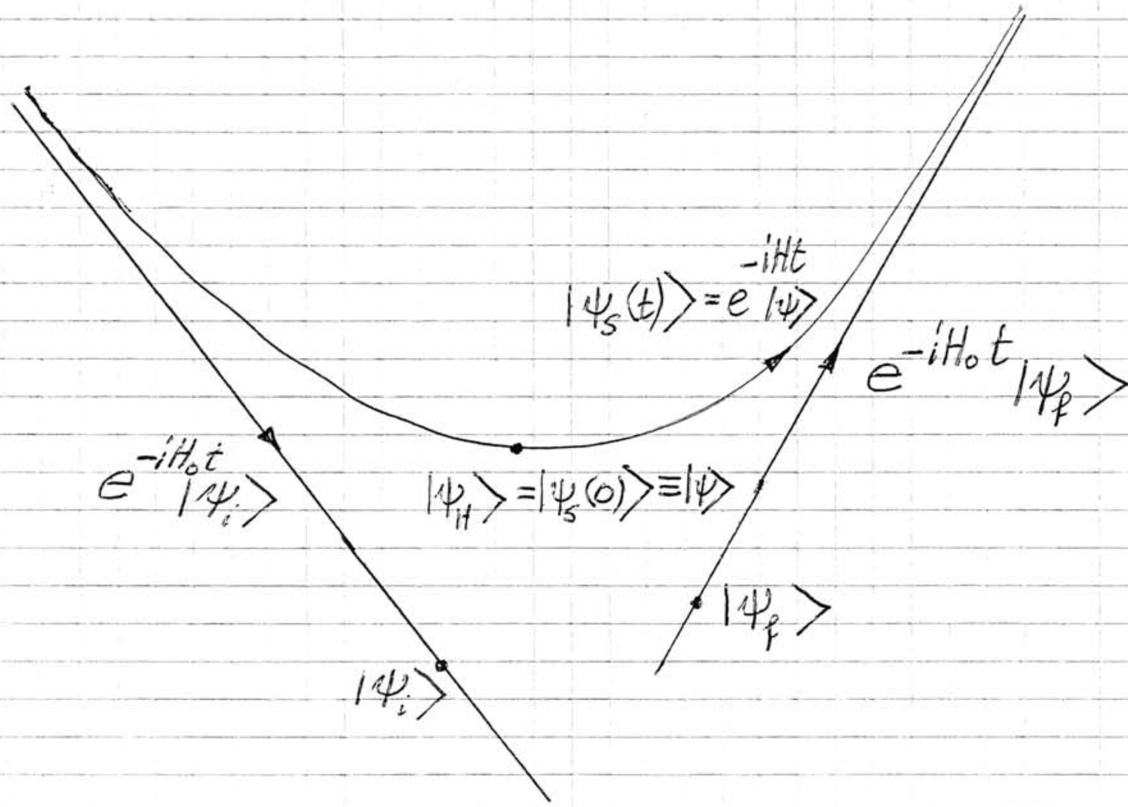
<u>note</u>: H_0 not necessarily free Hamiltonian

$$\langle 0 | T e^{-i\int_{-\infty}^{+\infty} dt\, H_1^{IP}(t)} | 0 \rangle = \langle 0 | T e^{i\int d^4x\, f(x)\phi(x)} | 0 \rangle$$

$= Z[f]$ ... generating of functional of ϕ^4-theory

(for $\lambda = 0$: of free scalar theory)

scattering theory



- free time-evolution symbolized by straight lines
- full time-evolution symbolized by curved line

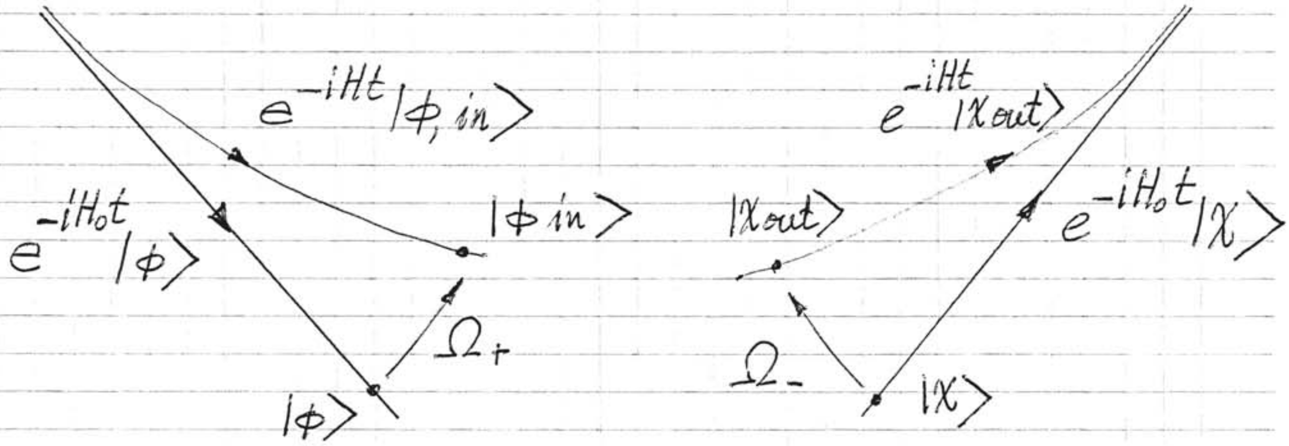
$$|\psi_s(t)\rangle = e^{-iHt}|\psi\rangle \xrightarrow[t\to-\infty]{} e^{-iH_0t}|\psi_i\rangle$$

$$|\psi_s(t)\rangle = e^{-iHt}|\psi\rangle \xrightarrow[t\to+\infty]{} e^{-iH_0t}|\psi_f\rangle$$

$$|\psi_{IP}(t)\rangle = e^{iH_0t}|\psi_s(t)\rangle \Rightarrow |\psi_{IP}(-\infty)\rangle = |\psi_i\rangle$$

$$|\psi_{IP}(+\infty)\rangle = |\psi_f\rangle$$

S-operator (S-matrix)



relevant (observable) quantity: $|\langle \chi\, out | \phi\, in\rangle|^2$

(S-matrix element $\langle \chi\, out | \phi\, in \rangle$)

$$|\phi\, in\rangle = \underbrace{\lim_{t\to-\infty} e^{iHt} e^{-iH_0 t}}_{\Omega_+} |\phi\rangle$$

$$|\chi\, out\rangle = \underbrace{\lim_{t\to+\infty} e^{iHt} e^{-iH_0 t}}_{\Omega_-} |\chi\rangle$$

$\Omega_\pm$ ... Møller operators

$$\langle \chi \text{ out} | \phi \text{ in} \rangle = \lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} \langle \chi | \underbrace{\underbrace{e^{iH_0 t} e^{-iHt}}_{U(t)} \underbrace{e^{iHt_0} e^{-iH_0 t_0}}_{U(t_0)^{-1}}}_{U(t,t_0)} | \phi \rangle$$

$$= \langle \chi | S | \phi \rangle$$

$$S := \lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} U(t,t_0) = \Omega_-^\dagger \Omega_+ =$$

$$= T e^{-i \int_{-\infty}^{+\infty} dt \, H_{int}^{IP}(t)} \underset{\substack{\uparrow \\ QFT}}{=} T e^{-i \int d^4x \, \mathcal{H}_{int}^{IP}(x)}$$

S-operator (in IP)

→ starting point of perturbative expansion in QFT