6. Abelian gauge field

gauge transformation $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x)$

gauge invariance of the action forbids mass term

$$\rightarrow \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\Rightarrow$ field equation $\quad (g_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\nu = 0$

inverse of differential operator $g_{\mu\nu} \Box - \partial_\mu \partial_\nu$ does not exist, as

$$(g_{\mu\nu} \Box - \partial_\mu \partial_\nu) \partial^\nu \Lambda = \Box \partial_\mu \Lambda - \partial_\mu \Box \Lambda = 0$$

remark: in momentum space,

$$T_{\mu\nu} = -g_{\mu\nu} k^2 + k_\mu k_\nu$$

$a = 1, \; b = -k^2 \;\Rightarrow\; ak^2 + b = 0$

$\Rightarrow T^{-1}$ does not exist

way out: gauge fixing term in the Lagrangean

(Fermi's trick)

$$\mathcal{L} \to \mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \underbrace{- \frac{\overset{\text{gauge parameter}}{\xi}}{2} (\partial_\mu A^\mu)^2}_{\text{gauge fixing term}} - J_\mu A^\mu$$

the action can now be written in the form

$$S = \int d^4x \left\{ \frac{1}{2} A^\mu \left[g_{\mu\nu} \Box - (1-\xi) \partial_\mu \partial_\nu \right] A^\nu - J_\mu A^\mu \right\}$$

the equation $[g_{\mu\nu} \Box - (1-\xi) \partial_\mu \partial_\nu] B^\nu = J_\mu$

can now be inverted:

$$T_{\mu\nu} = -k^2 g_{\mu\nu} + (1-\xi) k_\mu k_\nu$$

$$a = 1-\xi \, , \quad b = -k^2$$

$$\Rightarrow \quad (T^{-1})^{\nu\rho} = -\frac{1}{k^2} \left[g^{\nu\rho} - \left(1 - \frac{1}{\xi}\right) \frac{k^\nu k^\rho}{k^2} \right]$$

Feynman gauge ($\xi = 1$) : $\quad (T^{-1})^{\nu\rho} = -\frac{g^{\nu\rho}}{k^2}$

$$D^{\nu\rho}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \underbrace{\frac{-1}{k^2+i\varepsilon}\left[g^{\nu\rho} - \left(1-\frac{1}{\xi}\right)\frac{k^\nu k^\rho}{k^2}\right]}_{\tilde{D}^{\nu\rho}(k)}$$

$$S_{eff} = \int d^4x \left\{ \frac{1}{2} A^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\,\partial_\mu \partial_\nu\right] A^\nu - J_\mu A^\mu \right\}$$

$$\overset{A_\mu = A'_\mu + B_\mu}{=} \int d^4x \left\{ \frac{1}{2} A'^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\,\partial_\mu \partial_\nu\right] A'^\nu \right.$$

$$+ A'^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\,\partial_\mu \partial_\nu\right] B^\nu - J_\mu A'^\mu$$

$$\left. + \frac{1}{2} B^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\,\partial_\mu \partial_\nu\right] B^\nu - J_\mu B^\mu \right\}$$

$$\left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\right] B^\nu = J_\mu$$

i.e. $\quad B^\mu(x) = \int d^4y\, D^{\mu\nu}(x-y)\, J_\nu(y)$

→ terms linear in A'^μ disappear

$$\Rightarrow \quad S_{\text{eff}} = \int d^4x \left\{ \frac{1}{2} A'^{\mu} \left[g_{\mu\nu} (\Box - i\varepsilon) - (1-\xi) \partial_\nu \partial_\nu \right] A'^{\nu} - \frac{1}{2} J_\mu B^\mu \right\}$$

$$= \int d^4x \, \frac{1}{2} A'^{\mu} \left[g_{\mu\nu} (\Box - i\varepsilon) - (1-\xi) \partial_\mu \partial_\nu \right] A'^{\nu}$$

$$- \frac{1}{2} \int d^4x \, d^4y \, D^{\mu\nu}(x-y) \, J_\mu(x) \, J_\nu(y)$$

$$\Rightarrow \quad Z[J] = e^{-\frac{i}{2} \int d^4x \, d^4y \, D^{\mu\nu}(x-y) J_\mu(x) J_\nu(y)}$$

remark: $Z[J]$ is independent of the gauge parameter ξ for a conserved current ($\partial^\mu J_\mu = 0$): to see this, we compute

$$\int d^4y \, D^{\mu\nu}(x-y) \, J_\nu(y)$$

and convince ourselves that this expression is independent of ξ if $\partial^\mu J_\mu = 0$

$$J_\nu(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, \tilde{J}_\nu(p)$$

$$\partial^\nu J_\nu(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, (-ip^\nu)\, \tilde{J}_\nu(p)$$

$$\partial^\nu J_\nu = 0 \iff p^\nu \tilde{J}_\nu(p) = 0$$

$$\Rightarrow \int d^4y\; D^{\mu\nu}(x-y)\, J_\nu(y) =$$

$$= \int d^4y \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{-1}{k^2+i\varepsilon} \left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2} \right]$$

$$\times \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, \tilde{J}_\nu(p)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{-1}{k^2+i\varepsilon} \left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2} \right] \tilde{J}_\nu(k)$$

$$\underset{\substack{\uparrow \\ k^\nu \tilde{J}_\nu(k)=0}}{=} \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{-g^{\mu\nu}}{k^2+i\varepsilon}\, \tilde{J}_\nu(k)$$

$$\Rightarrow \int d^4x\, d^4y\; D^{\mu\nu}(x-y)\, J_\mu(x)\, J_\nu(y)$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{-\tilde{J}^\mu(k)}{k^2+i\varepsilon} \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \tilde{J}_\mu(p)$$

$$= -\int \frac{d^4k}{(2\pi)^4}\, \frac{\tilde{J}^\mu(k)\, \tilde{J}_\mu(-k)}{k^2+i\varepsilon}$$

$J^\mu(x)$ real $\Rightarrow \quad \tilde{J}^\mu(-k) = \tilde{J}^\mu(k)^*$

$$\Rightarrow \quad \int d^4x\, d^4y\; D^{\mu\nu}(x-y)\, J_\mu(x)\, J_\nu(y)$$

$$= -\int \frac{d^4k}{(2\pi)^4}\, \frac{\tilde{J}^\mu(k)\, \tilde{J}_\mu(k)^*}{k^2+i\varepsilon}$$

$$\Rightarrow \quad Z[J] = \langle 0\,\text{out} | 0\,\text{in} \rangle_J = e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}^\mu(k)\, \tilde{J}_\mu(k)^*}{k^2+i\varepsilon}}$$

Gupta-Bleuler quantization

field equation $\left(g_{\mu\nu}\Box - (1-\xi)\partial_\mu\partial_\nu\right)A^\nu = j_\mu$

in the classical theory one can impose the Lorenz gauge condition $\partial_\mu A^\mu = 0$

$\Rightarrow$ $\Box A^\mu = j^\mu$, $\partial_\mu A^\mu = 0$ are equivalent to Maxwell's equations

interpretation of $\partial_\mu A^\mu = 0$ in QFT?

consider free theory:

$$A_\mu(x) = \int d\mu(k)\,\left[e^{-ikx}a_\mu(k) + e^{+ikx}a_\mu(k)^\dagger\right]$$

canonical quantization (for $\xi = 1$):

$$[A_\mu(x), \pi_\nu(y)]\Big|_{x^0=y^0} = i\,g_{\mu\nu}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[A_\mu(x), A_\nu(y)]\Big|_{x^0=y^0} = [\pi_\mu(x), \pi_\nu(y)]\Big|_{x^0=y^0} = 0$$

$$\Leftrightarrow\quad [a_\mu(k), a_\nu(k')^\dagger] = -g_{\mu\nu}\,\delta(k,k')$$

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \Rightarrow \pi_0 = - \partial_\mu A^\mu, \; \pi_i = - F_{0i}$$

$\Rightarrow$ $\partial_\mu A^\mu = 0$ cannot be realized as an operator condition ($\pi_0 = 0$ would be in conflict with canonical quantization)

construction of Hilbert space of free photon states: vacuum state $|0\rangle$ defined by

$$a_\mu(\vec{k})\,|0\rangle = 0 \quad \forall \mu, \vec{k}$$

$$\langle 0|0\rangle = 1$$

photon states with given momentum $\vec{k}$:

$$|k, \varepsilon\rangle = - \varepsilon^\mu a_\mu(k)^\dagger |0\rangle$$

$\rightarrow$ four linear independent one-photon states with momentum $\vec{k}$ (instead of two)

timelike photon state: $\int d\mu(k)\, f_0(k)\, a_0(k)^\dagger |0\rangle$

scalar product :

$$\int d\mu(k)\, d\mu(k')\; f_0(k')^* f_0(k) \underbrace{\langle 0 | a_0(k')\, a_0^\dagger(k) | 0 \rangle}_{-\delta(k,k')}$$

$$= - \int d\mu(k)\, |f_0(k)|^2 < 0$$

<u>indefinite</u> metric

Gupta, Bleuler : <u>physical</u> states $|\psi\rangle$ satisfy the condition

$$\partial_\mu A^{\mu\,(+)} |\psi\rangle = 0$$

$$A_\mu^{(+)}(x) = \int d\mu(k)\, e^{-ikx}\, a_\mu(k)$$

this condition guarentees

$$\langle \psi | \partial_\mu A^\mu(x) | \psi \rangle = 0$$

for physical states $|\psi\rangle$

Gupta-Bleuler condition equivalent to

$$k^\mu a_\mu(k)\,|\psi\rangle = 0 \qquad \forall\, \vec{k}$$

remark: also valid in interacting theory as $\partial_\mu A^\mu$ behaves as a free field:

conserved current

$$\partial^\mu \,\Big|\, (g_{\mu\nu}\,\Box - (1-\xi)\,\partial_\mu\,\partial_\nu)\,A^\nu = j_\mu$$

$$\Rightarrow \quad \xi\,\Box\,\partial_\nu A^\nu = 0$$

$\langle\psi|\psi\rangle \geq 0$ for <u>physical</u> states $|\psi\rangle$

to see this we construct a new basis of creation and annihilation operators

$\vec{k}$ given $\to$ choose two unit vectors $\vec{\varepsilon}(k,1)$, $\vec{\varepsilon}(k,2)$ orthogonal to $\vec{k}$ $\Rightarrow$

$\vec{\varepsilon}(k,1)$, $\vec{\varepsilon}(k,2)$, $\vec{\varepsilon}(k,3) = \vec{k}/|\vec{k}|$ orthonormal basis in three-dim. space $\quad (\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}(k,\lambda') = \delta_{\lambda\lambda'})$

we define now:

$$a(k,1)^\dagger = \vec{\varepsilon}(k,1) \cdot \vec{a}(k)^\dagger$$

$$a(k,2)^\dagger = \vec{\varepsilon}(k,2) \cdot \vec{a}(k)^\dagger$$

$$b(k)^\dagger = \frac{1}{\sqrt{2}} \left(a_0(k)^\dagger - \vec{k} \cdot \vec{a}(k)^\dagger \right)$$

$$c(k)^\dagger = \frac{1}{\sqrt{2}} \left(a_0(k)^\dagger + \vec{k} \cdot \vec{a}(k)^\dagger \right)$$

$\Rightarrow$ commutation relations

$$[a(k,1), a(k',1)^\dagger] = [a(k,2), a(k',2)^\dagger] = \delta(k,k')$$

$$[b(k), b(k')^\dagger] = [c(k), c(k')^\dagger] = 0$$

$$[b(k), c(k')^\dagger] = [c(k), b(k')^\dagger] = -\delta(k,k')$$

(all other commutators vanish)

Gupta-Bleuler condition reads now:

$$b(k)\,|\psi\rangle = 0$$

for physical states $|\psi\rangle$

$$a(k,1)^\dagger \dots\ a(l,2)^\dagger \dots\ b(p)^\dagger \dots\ c(q)^\dagger \dots\ |0\rangle$$

Gupta-Bleuler condition only fulfilled if <u>no</u> $c^\dagger$ appears

example: $a(k,1)^\dagger|0\rangle$, $a(k,2)^\dagger|0\rangle$, $b(k)^\dagger|0\rangle$ satisfy the Gupta-Bleuler condition

$$\langle 0|\, a(k,1)\, a(k',1)^\dagger\, |0\rangle = \delta(k,k')$$

$$\langle 0|\, a(k,2)\, a(k',2)\, |0\rangle = \delta(k,k')$$

$$\langle 0|\, b(k)\, b(k')^\dagger\, |0\rangle = 0$$

construction of Fock space of physical states with positive definite metric: two physical vectors $|\psi_1\rangle$, $|\psi_2\rangle$ are <u>equivalent</u> ($|\psi_1\rangle \sim |\psi_2\rangle$) if

$$\langle \psi_1 - \psi_2 \,|\, \psi_1 - \psi_2 \rangle = 0$$

$$\Rightarrow \quad b(k)^{\dagger}|0\rangle \sim \emptyset$$

only $a(k,1)^{\dagger}|0\rangle$ and $a(k,2)^{\dagger}|0\rangle$

are one-photon states $\neq \emptyset$