6. Abelian gauge field

gauge transformation \( A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \)
gauge invariance of the action forbids mass term

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ \Rightarrow \text{field equation} \quad (g_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\nu = 0 \]

inverse of differential operator \( g_{\mu\nu} \Box - \partial_\mu \partial_\nu \)
does not exist, as

\[ (g_{\mu\nu} \Box - \partial_\mu \partial_\nu) \Box \Lambda = \Box \partial_\mu \Lambda - \partial_\mu \Box \Lambda = 0 \]

remark: in momentum space,

\[ T_{\mu\nu} = -g_{\mu\nu} k^2 + k_\mu k_\nu \]

\[ a = 1, \quad b = -k^2 \Rightarrow ak^2 + b = 0 \]

\[ \Rightarrow T^{-1} \text{ does not exist} \]
way out: gauge fixing term in the Lagrangean

\[ \mathcal{L} \rightarrow \mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\xi}{2} (\partial_{\mu} A^{\mu})^2 - J_{\mu} A^{\mu} \]

gauge parameter
gauge fixing term

the action can now be written in the form

\[ S = \int d^4x \left\{ \frac{1}{2} F^{\mu\nu} \left[ g_{\mu\nu} \Box - (1-\xi) \partial_{\mu} \partial_{\nu} \right] A^{\nu} \right\} \]

\[ - J_{\mu} A^{\mu} \}

the equation \[ \left[ g_{\mu\nu} \Box - (1-\xi) \partial_{\mu} \partial_{\nu} \right] B^{\nu} = J_{\mu} \]
can now be inverted:

\[ T_{\mu\nu} = - R^2 g_{\mu\nu} + (1-\xi) k_{\mu} k_{\nu} \]

\[ a = 1 - \frac{\xi}{2}, \quad b = -R^2 \]

\[ \Rightarrow (T-1)^{\nu\delta} = -\frac{1}{R^2} \left[ g^{\nu\delta} - (1-\xi) \frac{k_{\nu} k_{\delta}}{R^2} \right] \]
Feynman gauge \((\xi = 1)\): \((T^{-1})^{\phi} = -\frac{\Phi^{\phi}}{R^2}\)

\[
\mathcal{D}^{\phi}(x) = \int \frac{d^4R}{(2\pi)^4} \ e^{-iR^2} \left[ \frac{1}{R^2 + i\epsilon} \right] \left[ g^{\phi} \phi - \left(1 - \frac{i}{\phi} \right) \frac{R^2 \phi R^2}{R^2} \right]
\]

\[
\tilde{\mathcal{D}}^{\phi}(R)
\]

\[
S_{\phi} = \int d^4x \left\{ \frac{i}{2} A^\mu \left[ g_{\mu\nu} (\Box - i\epsilon) - (1 - \xi) \partial_\mu \partial_\nu \right] A^\nu - \partial_\mu A^\mu \right\}
\]

\[
A^\mu = A'^\mu + B^\mu
\]

\[
\Downarrow
\]

\[
\int d^4x \left\{ \frac{i}{2} A'^\mu \left[ g_{\mu\nu} (\Box - i\epsilon) - (1 - \xi) \partial_\mu \partial_\nu \right] A'^\nu
\]

\[
+ A'^\mu \left[ g_{\mu\nu} (\Box - i\epsilon) - (1 - \xi) \partial_\mu \partial_\nu \right] B^\nu - \partial_\mu A'^\mu
\]

\[
+ \frac{i}{2} B^\mu \left[ g_{\mu\nu} (\Box - i\epsilon) - (1 - \xi) \partial_\mu \partial_\nu \right] B^\nu - \partial_\mu B^\mu
\}

\[
\left[ g_{\mu\nu} (\Box - i\epsilon) - (1 - \xi) \right] B^\nu = \partial_\mu
\]

\[
i.e. \quad B^\mu(x) = \int d^4y \mathcal{D}^{\mu\nu}(x-y) \ J^\nu(y)
\]

\[
\rightarrow \text{terms linear in } A'^\mu \text{ disappear}
\]
\[ S_{\text{eff}} = \int d^4x \left\{ \frac{1}{2} A^\mu \left[ g_{\mu\nu} (\partial - i\xi) - (1 - \xi) \partial_\mu \partial_\nu \right] A^\nu \\
- \frac{1}{2} J^\mu B^\mu \right\} \]

\[ = \int d^4x \left( \frac{i}{2} A^\mu \left[ g_{\mu\nu} (\partial - i\xi) - (1 - \xi) \partial_\mu \partial_\nu \right] A^\nu \right) \]

\[ - \frac{1}{2} \int d^4x \int d^4y \; D^{\mu\nu}(x-y) J_\mu(x) J_\nu(y) \]

\[ \Rightarrow \tilde{Z} [J] = e^{-\frac{i}{2} \int d^4x \int d^4y \; D^{\mu\nu}(x-y) J_\mu(x) J_\nu(y)} \]

Remark: \( \tilde{Z} [J] \) is independent of the gauge parameter \( \xi \) for a conserved current (\( \partial^\mu J_\mu = 0 \)).

To see this, we compute

\[ \int d^4y \; D^{\mu\nu}(x-y) J_\nu(y) \]

and convince ourselves that this expression is independent of \( \xi \) if \( \partial^\mu J_\mu = 0 \).
\[ J_{\nu}(y) = \int \frac{d^4 p}{(2\pi)^4} \ e^{-ip \cdot y} \tilde{J}_{\nu}(p) \]

\[ \partial_\rho \tilde{J}_{\nu}(y) = \int \frac{d^4 p}{(2\pi)^4} \ e^{-ip \cdot y} (-i \rho^\nu) \tilde{J}_{\nu}(p) \]

\( \partial_\rho \tilde{J}_{\nu} = 0 \iff p^\rho \tilde{J}_{\nu}(p) = 0 \)

\[ \Rightarrow \int d^4 y \ D^{\rho \nu}(x-y) \tilde{J}_{\nu}(y) = \]

\[ = \int d^4 y \int \frac{d^4 p}{(2\pi)^4} \ e^{-iK(x-y)} \frac{1}{R^2 + i\varepsilon} \left[g^{\mu \nu} - (1 - \frac{1}{3}) \frac{R^\mu R^\nu}{R^2} \right]
\]

\[ \times \int \frac{d^4 p}{(2\pi)^4} \ e^{-ip \cdot y} \tilde{J}_{\nu}(p) \]

\[ = \int \frac{d^4 K}{(2\pi)^4} \ e^{-i K x} \frac{1}{R^2 + i\varepsilon} \left[g^{\mu \nu} - (1 - \frac{1}{3}) \frac{R^\mu R^\nu}{R^2} \right] \tilde{J}_{\nu}(K) \]

\[ \Rightarrow \int \frac{d^4 K}{(2\pi)^4} \ e^{-i K x} \frac{-g^{\mu \nu}}{R^2 + i\varepsilon} \tilde{J}_{\nu}(K) \]

\[ \Rightarrow \rho^\nu \tilde{J}_{\nu}(R) = 0 \]
\[ \Rightarrow \int d^4x \, d^4y \, D^{\mu\nu}(x-y) \, J_\mu(x) \, J_\nu(y) \]

\[ = \int d^4x \, \frac{d^4k}{(2\pi)^4} \, e^{-ikx} \, \frac{\tilde{J}^\mu(k)}{k^2 + i\varepsilon} \, \int d^4p \, \frac{d^4p'}{(2\pi)^4} \, e^{-ipx'} \, \tilde{J}^\nu(p') \]

\[ = -\frac{\tilde{J}^\mu(k)}{k^2 + i\varepsilon} \cdot \frac{\tilde{J}^\nu(-k)}{k^2 + i\varepsilon} \]

\[ \tilde{J}^\mu(x) \text{ real} \Rightarrow \tilde{J}^\mu(-k) = \tilde{J}^\mu(k)^* \]

\[ \Rightarrow \int d^4x \, d^4y \, D^{\mu\nu}(x-y) \, J_\mu(x) \, J_\nu(y) \]

\[ = -\frac{\tilde{J}^\mu(k)}{k^2 + i\varepsilon} \cdot \frac{\tilde{J}^\nu(k)^*}{k^2 + i\varepsilon} \]

\[ \Rightarrow \mathcal{Z}[J] = \left< \text{out} \left| \text{in} \right. \right> = e^{\frac{i}{2} \int d^4x \, \frac{\tilde{J}^\mu(k)}{k^2 + i\varepsilon} \cdot \frac{\tilde{J}^\nu(k)^*}{k^2 + i\varepsilon} } \]
Gupta-Blumler quantization

Field equation

\[ (g^{\mu \nu} \Box - (1 - \xi) \partial_\mu \partial_\nu) A^\nu = j_\mu \]

in the classical theory one can impose the Lorentz gauge condition \( \partial_\mu A^\mu = 0 \)

\[ \Rightarrow \Box A^\mu = j_\mu, \quad \partial_\mu A^\mu = 0 \] are equivalent to Maxwell's equations

interpretation of \( \partial_\mu A^\mu = 0 \) in QFT:

consider free theory:

\[ A_\mu (x) = \int dq (k) [e^{-i k x} a_\mu (k) + e^{i k x} a_\mu^\dagger (k)] \]

canonical quantization (for \( \xi = 1 \)):

\[ [A_\mu (x), \pi_\nu (y)] \bigg|_{x^0 = y^0} = i g_{\mu \nu} \delta^{(3)} (x-y) \]

\[ [A_\mu (x), A_\nu (y)] \bigg|_{x^0 = y^0} = [\pi_\mu (x), \pi_\nu (y)] \bigg|_{x^0 = y^0} = 0 \]

\[ \Rightarrow [a_\mu (k), a_\nu (k')^\dagger] = -g_{\mu \nu} \delta (k, -k') \]
\[ \pi_\mu = \frac{\partial L}{\partial \dot{A}_\mu} \Rightarrow \pi_0 = - \partial_\mu A^\mu, \pi_i = - F_{0i} \]

\[ \Rightarrow \partial_\mu A^\mu = 0 \text{ cannot be realized as an operator condition (} \pi_0 = 0 \text{ would be in conflict with canonical quantization)} \]

Construction of Hilbert space of free photon states: vacuum state \( |0\rangle \) defined by

\[ \epsilon_\mu (\mathbf{k}) |0\rangle = 0 \quad \forall \mu, \mathbf{k} \]

\[ \langle 0 | 0 \rangle = 1 \]

Photon states with given momentum \( \mathbf{k} \):

\[ |\mathbf{k}, \varepsilon \rangle = - \varepsilon^\dagger \epsilon_\mu (\mathbf{k})^\dagger |0\rangle \]

→ four linear independent one-photon states with momentum \( \mathbf{k} \) (instead of two)

Timelike photon state: \( \int d\mu (\mathbf{k}) f_\mu (\mathbf{k}) A_\mu (\mathbf{k})^\dagger |10\rangle \)
scalar product:

$$\int d\mu(k) d\mu(k') f_0(k')^* f_0(k) <0| \alpha_0(k') \alpha_0^+(k) > - \delta(k, k')$$

$$= - \int d\mu(k) |f_0(k)|^2 <0$$

indefinite metric

Gupta, Böeuler: physical states $|\psi\rangle$ satisfy the condition

$$\partial_\mu A_\mu^+(x) |\psi\rangle = 0$$

$$A_\mu^+(x) = \int d\mu(k) e^{-ikx} \alpha_\mu(k)$$

this condition guarantees

$$\langle \psi | \partial_\mu A_\mu^+(x) |\psi\rangle = 0$$

for physical states $|\psi\rangle$
Gupta-Blauer condition equivalent to

\[ R^\mu \partial_\mu (k) |\psi\rangle = 0 \quad \forall \vec{k} \]

remark: also valid in interacting theory as
\( \partial_\mu A^\mu \) behaves as a free field:

\[ \partial_\mu \left[ (g_{\mu\nu} \Box - (1-\xi) \partial_\mu \partial_\nu) A^\nu \right] = j_\mu \]

\[ \Rightarrow \xi \Box \partial_\nu A^\nu = 0 \]

\[ \langle \psi | \psi \rangle \geq 0 \text{ for physical states } |\psi\rangle \]

to see this we construct a new basis of
creation and annihilation operators
\( \vec{k} \): given \(-\) choose two unit vectors \( \vec{e}(k,1), \vec{e}(k,2) \) orthogonal to \( \vec{k} \) \(-\)
\( \vec{e}(k,1), \vec{e}(k,2), \vec{e}(k,3) = \vec{k}/|\vec{k}| \) orthonormal
basis in three-dim. space (\( \vec{e}(k,x). \vec{e}(k,x') = \delta_{xx'} \)
we define now:

\[ a(R,1)^\dagger = \vec{e}(R,1) \cdot \vec{a}(R)^\dagger \]
\[ a(R,2)^\dagger = \vec{e}(R,2) \cdot \vec{a}(R)^\dagger \]
\[ b(R)^\dagger = \frac{1}{12} \left( a_0(R)^\dagger - \vec{R} \cdot \vec{a}(R)^\dagger \right) \]
\[ c(R)^\dagger = \frac{1}{12} \left( a_0(R)^\dagger + \vec{R} \cdot \vec{a}(R)^\dagger \right) \]

⇒ commutation relations

\[ [a(R,1), a(R',1)^\dagger] = [a(R,2), a(R',2)^\dagger] = \delta(R,R') \]
\[ [b(R), b(R')^\dagger] = [c(R), c(R')^\dagger] = 0 \]
\[ [b(R), c(R')^\dagger] = [c(R), b(R')^\dagger] = -\delta(R,R') \]

(all other commutators vanish)

Gupta–Bleuler condition reads now:

\[ b(R) |\psi\rangle = 0 \]

for physical states |\psi\rangle
\[ a(k,1)^t \ldots a(l,2)^t \ldots b(p)^t \ldots c(q)^t \ldots |0\rangle \]

Gupta-Bleuler condition only fulfilled if no \( c^t \) appears.

Example: \( a(k,1)^t |0\rangle \), \( a(k,2)^t |0\rangle \), \( b(R)^t |0\rangle \)

satisfy the Gupta-Bleuler condition

\[ \langle 0 | a(k,1) a(k',1)^t |0\rangle = \delta(k, k') \]
\[ \langle 0 | a(k,2) a(k',2)^t |0\rangle = \delta(k, k') \]
\[ \langle 0 | b(R) b(R')^t |0\rangle = 0 \]

construction of Fock space of physical states with positive definite metric: two physical vectors \( |\Psi_1\rangle \), \( |\Psi_2\rangle \) are equivalent if

\[ \langle \Psi_1 - \Psi_2 | \Psi_1 - \Psi_2 \rangle = 0 \]
\[ \Rightarrow \quad b(k)^\dagger |0\rangle \sim \emptyset \]

Only \[ a(k,1)^\dagger |0\rangle \text{ and } a(k,2)^\dagger |0\rangle \]

are one-photon states \( \neq \emptyset \)