5. Massive vector field

$V_\mu(x)$ describes massive spin 1 field (vector boson)

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2}{2} V_\mu V^\mu$$

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \quad , \quad V_\mu \text{ real}$$

$\Rightarrow$ field equation $(\Box + M^2) V^\mu - \partial^\mu \partial_\nu V^\nu = 0$

$$\Rightarrow \quad (\Box + M^2) \partial_\mu V^\mu - \Box \partial_\nu V^\nu = M^2 \partial_\mu V^\mu = 0$$

$\overset{M \neq 0}{\Rightarrow} \quad \partial_\mu V^\mu = 0$ (projects out spin 0 component)

$$\boxed{\begin{aligned} (\Box + M^2) V^\mu &= 0 \\ \partial_\mu V^\mu &= 0 \end{aligned}}$$

plane wave solutions :

$$\varepsilon^\mu e^{\pm ikx} \quad , \quad k^2 = M^2 \, , \quad k_\mu \varepsilon^\mu = 0$$

rest frame: $k = \begin{pmatrix} M \\ \vec{0} \end{pmatrix} \Rightarrow \varepsilon = \begin{pmatrix} 0 \\ \vec{\varepsilon} \end{pmatrix}$

normalization $|\vec{\varepsilon}| = 1 \Rightarrow \varepsilon^2 = -1$

3 polarizations for given $\vec{k}$: $\varepsilon^\mu(k,\lambda)$, $\lambda = 1,2,3$

$$\varepsilon(k,1) = \begin{pmatrix} 0 \\ \vec{\varepsilon}(k,1) \end{pmatrix} , \quad \varepsilon(k,2) = \begin{pmatrix} 0 \\ \vec{\varepsilon}(k,2) \end{pmatrix}$$

$$\vec{\varepsilon}(k,1) \cdot \vec{k} = \vec{\varepsilon}(k,2) \cdot \vec{k} = 0 \qquad \textit{transversal}$$

$$\varepsilon(k,3) = \frac{1}{M} \begin{pmatrix} |\vec{k}| \\ \frac{k^0 \vec{k}}{|\vec{k}|} \end{pmatrix} \qquad \textit{longitudinal}$$

$$\varepsilon^\mu(k,\lambda)\, \varepsilon_\mu(k,\sigma) = -\delta_{\lambda\sigma} , \quad \varepsilon^\mu(k,\lambda)\, k_\mu = 0$$

$$\sum_\lambda \varepsilon^\mu(k,\lambda)\, \varepsilon^\nu(k,\lambda) = -g^{\mu\nu} + k^\mu k^\nu / M^2$$

$\longrightarrow$ general solution of field equations:

$$V^\mu(x) = \sum_{\lambda=1}^{3} \int d\mu(k) \left[\varepsilon^\mu(k,\lambda)\, a(k,\lambda)\, e^{-ikx} + h.c. \right]$$

quantization: $[a(k,\lambda), a(k',\lambda')^\dagger] = \delta_{\lambda\lambda'}\, \delta(k,k')$

<u>generating functional</u> of free massive vector field (using the functional integral method):

$$Z[J] = \langle 0 | T\, e^{-i\int d^4x\, V^\mu(x) J_\mu(x)} | 0 \rangle$$

$$= \frac{1}{\mathcal{N}} \int [dV^\mu]\, e^{i\int d^4x \left(-\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2 - i\varepsilon}{2} V_\mu V^\mu - V^\mu J_\mu\right)}$$

external current $J_\mu(x)$, normalization $Z[0] = 1$

$$S = \int d^4x \left(-\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2 - i\varepsilon}{2} V_\mu V^\mu - J_\mu V^\mu\right)$$

$$= \int d^4x \left\{ \frac{1}{2} V^\mu \left[g_{\mu\nu} (\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu \right] V^\nu - J_\mu V^\mu \right\}$$

usual trick: shift of integration variable

$$V^\mu = \underset{\uparrow}{V'^\mu} + W^\mu$$

new integration variable in the functional integral

$[dV^\mu] = [dV'^\mu]$ translation invariance of the measure terms linear in V'^{μ} disappear, if

$$[g_{\mu\nu}(\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu] W'^{\nu} = J_\mu$$

propagator $\Delta^{\nu\rho}(x)$ = Green's function of the differential operator $g_{\mu\nu}(\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu$:

$$[g_{\mu\nu}(\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu] \Delta^{\nu\rho}(x) = \underbrace{\delta_\mu{}^\rho}_{g_\mu{}^\rho} \delta^{(4)}(x)$$

Fourier representation $\Delta^{\nu\rho}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\Delta}^{\nu\rho}(k)$

$\longrightarrow$ equation in momentum space

$$[g_{\mu\nu}(-k^2 + M^2 - i\varepsilon) + k_\mu k_\nu] \tilde{\Delta}^{\nu\rho}(k) = g_\mu{}^\rho$$

we discuss the more general problem:

find the inverse of

$$T_{\mu\nu} = a(k^2)\, k_\mu k_\nu + b(k^2)\, g_{\mu\nu}$$

ansatz for $(T^{-1})^{\nu\rho}$:

$$(T^{-1})^{\nu\rho} = A(k^2)\, k^\nu k^\rho + B(k^2)\, g^{\nu\rho}$$

$$(a\, k_\mu k_\nu + b\, g_{\mu\nu})(A\, k^\nu k^\rho + B\, g^{\nu\rho}) = \delta_\mu{}^\rho$$

$$a A\, k^2 k_\mu k^\rho + a B\, k_\mu k^\rho + b A\, k_\mu k^\rho$$
$$+ b B\, g_\mu{}^\rho = \delta_\mu{}^\rho$$

$$\Rightarrow \; B = \frac{1}{b}\,, \quad a A k^2 + a B + b A = 0$$

$$A\,(a k^2 + b) = -\frac{a}{b} \;\rightarrow\; A = -\frac{a}{b(a k^2 + b)}$$

$$\Rightarrow (T^{-1})^{\nu\rho} = \frac{g^{\nu\rho}}{b} - \frac{a\, k^\nu k^\rho}{b\,(a k^2 + b)}$$

$\Rightarrow$ T^{-1} exists if $b \neq 0$ and $a k^2 + b \neq 0$

in the case of the massive vector field we have

$a = 1\,, \quad b = -k^2 + M^2 - i\varepsilon$

$$\Rightarrow \quad \tilde{\Delta}^{\nu\rho}(k) = \frac{g^{\nu\rho} - k^\nu k^\rho / M^2}{M^2 - k^2 - i\varepsilon}$$

$$\Delta^{\nu\rho}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{g^{\nu\rho} - k^\nu k^\rho / M^2}{M^2 - k^2 - i\varepsilon}$$

$$W^\mu(x) = \int d^4y \; \Delta^{\mu\nu}(x-y)\, J_\nu(y)$$

$$Z[J] = e^{-\frac{i}{2}\int d^4x\, J_\mu(x)\, W^\mu(x)}$$

$$= e^{-\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\, \Delta^{\mu\nu}(x-y)\, J_\nu(y)}$$

$$\Rightarrow \quad -\frac{1}{2} \langle 0 | T\, V^\mu(x)\, V^\nu(y) | 0 \rangle = -\frac{i}{2}\, \Delta^{\mu\nu}(x-y)$$

$$\langle 0 | T\, V^\mu(x)\, V^\nu(y) | 0 \rangle = i\, \Delta^{\mu\nu}(x-y)$$

$$= i \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{g^{\mu\nu} - k^\mu k^\nu / M^2}{M^2 - k^2 - i\varepsilon}$$

Complex vector field ($M \neq 0$)

$$\mathcal{L} = -\frac{1}{2} V_{\mu\nu}^{*} V^{\mu\nu} + M^2 V_\mu^* V^\mu$$

$$V^\mu = (V_1^\mu + i V_2^\mu)/\sqrt{2} \quad , \quad V_{1,2}^\mu \text{ real}$$

generating functional

$$Z[J, J^*] = \langle 0 | T \, e^{-i\int d^4x \, (V_\mu^\dagger(x) J^\mu(x) + V^\mu(x) J_\mu^*(x))} | 0 \rangle$$

can be obtained from the generating functional of a real vector field $\left(J^\mu = (J_1^\mu + i J_2^\mu)/\sqrt{2} , \right.$

$\left. J_1, J_2 \text{ real} \right)$

$$\rightarrow \quad Z[J, J^*] = e^{-i\int d^4x \, d^4y \, J_\mu^*(x) \Delta^{\mu\nu}(x-y) J_\nu(y)}$$

$$\Rightarrow \quad \langle 0 | T \, V_\mu(x) \, V_\nu^\dagger(y) | 0 \rangle = i \, \Delta^{\mu\nu}(x-y)$$

$$\langle 0 | T \, V_\mu(x) \, V_\nu(y) | 0 \rangle = 0$$