# 4. Fermionic path integral

in the case of bosonic variables, the Gaussian mean-value could be interpreted as the limit of (ordinary) integrals of the form

$$\left\langle\!\left\langle \mathcal{F}(\varphi) \right\rangle\!\right\rangle = \frac{\int d\varphi_1 \dots d\varphi_N \, e^{-\frac{1}{2}\varphi^T A \varphi} \, \mathcal{F}(\varphi)}{\int d\varphi_1 \dots d\varphi_N \, e^{-\frac{1}{2}\varphi^T A \varphi}}$$

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix}, \quad A^T = A$$

$$\left\langle\!\left\langle \varphi_{k_1} \dots \varphi_{k_n} \right\rangle\!\right\rangle = \left\langle\!\left\langle \varphi_{k_{\sigma(1)}} \dots \varphi_{k_{\sigma(n)}} \right\rangle\!\right\rangle \qquad \sigma \in S_n$$

$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum_{\text{pairings}} \left\langle\!\left\langle \varphi_{k_1'} \varphi_{k_2'} \right\rangle\!\right\rangle \dots \left\langle\!\left\langle \varphi_{k_{n-1}'} \varphi_{k_n'} \right\rangle\!\right\rangle & \text{if } n \text{ even} \end{cases}$$

fermionic variables

anticommuting objects $\chi_1, \dots, \chi_N$

$$\{\chi_k, \chi_l\} = 0$$

elements of a <u>Grassmann algebra</u>

Gaussian mean value of Grassmann variables:

$$\langle\!\langle \chi_{k_1} \ldots \chi_{k_n} \rangle\!\rangle = (-1)^{\sigma} \langle\!\langle \chi_{k_{\sigma(1)}} \ldots \chi_{k_{\sigma(n)}} \rangle\!\rangle \quad \sigma \in S_n$$

$$(-1)^{\sigma} = \begin{cases} 1 & \text{if } \sigma \text{ even permutation of } 1, \ldots, n \\ -1 & \text{〃 〃 odd 〃 〃 〃} \end{cases}$$

$$\langle\!\langle \chi_{k_1} \ldots \chi_{k_n} \rangle\!\rangle = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum\limits_{\text{pairings}} (-1)^{\sigma} \langle\!\langle \chi_{k_1'} \chi_{k_2'} \rangle\!\rangle \ldots \langle\!\langle \chi_{k_{n-1}'} \chi_{k_n'} \rangle\!\rangle & \end{cases}$$

$$(-1)^{\sigma} = \begin{cases} +1 & \text{if } k_1', \ldots, k_n' \text{ even permutation of} \\ -1 & \text{〃 〃 odd 〃 〃 } k_1, \ldots, k_n \end{cases}$$

Gaussian mean value of fermionic variables <u>cannot</u> be represented by ordinary integrals but by "integrals"

$$\int \underbrace{d\chi_1 \ldots d\chi_N}_{[d\chi]} f(\chi_1, \ldots, \chi_N)$$

over the Grassmann variables $\chi_1, \ldots, \chi_N$, which are <u>not</u> defined by a limiting process (like the Riemann- or Lebesgue-integral) but are defined implicitly in terms of their properties (essential property: integration measure is <u>translation invariant</u> $\Rightarrow \int d\chi = 0 \,, \int d\chi\, \chi = \text{const.} (=1)$

$\uparrow$ single variable

we consider the integral

$$I = \int [d\chi] \exp\left(-\frac{1}{2} D^{k\ell} \chi_k \chi_\ell + f^k \chi_k\right)$$

$D^{k\ell} = -D^{\ell k}$ is an anti-symmetric matrix, the $f^k$ are also Grassmann variables:

$$\{\chi_k, f^\ell\} = \{f^k, f^\ell\} = 0$$

change of variables $\chi_k = \chi_k' + c_k \quad [d\chi] = [d\chi']$

$$I = \int [d\chi'] \exp\left(-\frac{1}{2} D^{k\ell} (\chi_k' + c_k)(\chi_\ell' + c_\ell) + f^k (\chi_k' + c_k)\right)$$

$$= \exp\left(-\frac{1}{2} D^{k\ell} c_k c_\ell + f^k c_k\right)$$

$$\times \int [d\chi'] \exp\left(-\frac{1}{2} D^{k\ell} \chi_k' \chi_\ell' - \frac{1}{2} D^{k\ell} \chi_k' c_\ell - \frac{1}{2} D^{k\ell} c_k \chi_\ell' + f^k \chi_k'\right.$$

we choose $c_k = - D^{-1}_{kl} f^l \Rightarrow$ terms linear in $\chi'$ disappear

$$\Rightarrow \quad I = \exp\left(-\tfrac{1}{2} D^{-1}_{kl} f^k f^l\right) \underbrace{\int [d\chi] \exp\left(-\tfrac{1}{2} D^{kl} \chi_k \chi_l\right)}_{I_0}$$

Gaussian mean value defined by

$$\langle\!\langle \mathcal{F}(\chi) \rangle\!\rangle = \frac{\int [d\chi] \exp\left(-\frac{1}{2} D^{kl} \chi_k \chi_l\right) \mathcal{F}(\chi)}{\int [d\chi] \exp\left(-\frac{1}{2} D^{kl} \chi_k \chi_l\right)}$$

$$\Rightarrow \quad \langle\!\langle \exp(f^k \chi_k) \rangle\!\rangle = \exp\left(-\tfrac{1}{2} D^{-1}_{kl} f^k f^l\right)$$

expansion of exponential on both sides $\Rightarrow$

$$1 + \langle\!\langle f^k \chi_k \rangle\!\rangle + \frac{1}{2!} \langle\!\langle \underbrace{f^k \chi_k \, f^l \chi_l}_{f^k f^l \chi_k \chi_l} \rangle\!\rangle + \ldots$$

$$= 1 - \tfrac{1}{2} D^{-1}_{kl} f^k f^l + \ldots$$

$$\Rightarrow \quad \langle\!\langle \chi_k \rangle\!\rangle = 0$$

$$\Rightarrow \quad \langle\!\langle \chi_k \chi_l \rangle\!\rangle = D^{-1}_{kl}$$

$\ldots \Rightarrow$ pairing rule (see above) *special case:*

$$D = \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix}, \quad \chi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad f = \begin{pmatrix} \bar{\eta} \\ -\eta \end{pmatrix}$$

$$-\frac{1}{2} D^{kl} \chi_k \chi_l + f^k \chi_k = -\frac{1}{2} \chi^T D \chi + f^T \chi =$$

$$= -\frac{1}{2} \left(\psi^T, \bar{\psi}^T\right) \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$+ \left(\bar{\eta}^T, -\eta^T\right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$= -\frac{1}{2} \left(\psi^T, \bar{\psi}^T\right) \begin{pmatrix} -B^T \bar{\psi} \\ B \psi \end{pmatrix} + \bar{\eta}^T \psi - \eta^T \bar{\psi}$$

$$= +\frac{1}{2} \psi_\alpha B^{\beta\alpha} \bar{\psi}_\beta - \frac{1}{2} \bar{\psi}_\alpha B^{\alpha\beta} \psi_\beta + \bar{\eta}^\alpha \psi_\alpha + \bar{\psi}_\alpha \eta^\alpha$$

$$= -\bar{\psi}_\alpha B^{\alpha\beta} \psi_\beta + \bar{\eta}^\alpha \psi_\alpha + \bar{\psi}_\alpha \eta^\alpha$$

$\psi_\alpha, \bar{\psi}_\alpha$ treated as independent variables; <u>no</u> (anti-) symmetry condition for $B^{\alpha\beta}$

$$D^{-1} = \begin{pmatrix} 0 & B^{-1} \\ -(B^T)^{-1} & 0 \end{pmatrix}$$

$$\Rightarrow \; -\frac{1}{2} f^k D^{-1}_{kl} f^l = -\frac{1}{2} \left(\bar{\eta}^T, -\eta^T\right) \begin{pmatrix} 0 & B^{-1} \\ -(B^T)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\eta} \\ -\eta \end{pmatrix}$$

$$= -\frac{1}{2} \left(\bar{\eta}^T, -\eta^T\right) \begin{pmatrix} -B^{-1}\eta \\ -(B^T)^{-1}\bar{\eta} \end{pmatrix}$$

$$= \frac{1}{2} \left[ \bar{\eta}^T B^{-1} \eta - \eta^T (B^T)^{-1} \bar{\eta} \right]$$

$$= \frac{1}{2} \left[ \bar{\eta}^\alpha B^{-1}_{\alpha\beta} \eta^\beta - \eta^\beta B^{-1}_{\alpha\beta} \bar{\eta}^\alpha \right]$$

$$= \bar{\eta}^\alpha B^{-1}_{\alpha\beta} \eta^\beta$$

$$\Rightarrow \; I = \int [d\psi \, d\bar{\psi}] \; \exp\left(-\bar{\psi}_\alpha B^{\alpha\beta} \psi_\beta + \bar{\eta}^\alpha \psi_\alpha + \bar{\psi}_\alpha \eta^\alpha\right)$$

$$= I_0 \; \exp\left(\bar{\eta}^\alpha B^{-1}_{\alpha\beta} \eta^\beta\right)$$

$$\Rightarrow \; \langle\!\langle \psi_\alpha \bar{\psi}_\beta \rangle\!\rangle = B^{-1}_{\alpha\beta}$$

$$\langle\!\langle \psi_\alpha \psi_\beta \rangle\!\rangle = \langle\!\langle \bar{\psi}_\alpha \bar{\psi}_\beta \rangle\!\rangle = 0$$

generating functional of the free Dirac field

$$Z[\eta, \bar{\eta}] =$$

field operators

$$= \langle 0 | T \exp\left\{ i \int d^4x \left[ \bar{\eta}(x) \Psi(x) + \bar{\Psi}(x) \eta(x) \right] \right\} | 0 \rangle$$

fermionic sources

$$= \frac{1}{\mathcal{N}} \int [d\psi \, d\bar{\psi}] \exp\left\{ i \int d^4x \left[ \bar{\psi}(x)(i\not{\partial} - m + i\varepsilon)\psi(x) + \bar{\eta}(x)\psi(x) + \right.\right.$$

$$\left.\left. + \bar{\psi}(x) \eta(x) \right] \right\}$$

$\psi, \bar{\psi}, \eta, \bar{\eta}$ Grassmann fields

$[d\psi \, d\bar{\psi}]$ translation-invariant measure

normalization $Z[0,0] = 1$

computation of $Z[\eta, \bar{\eta}]$ :

shift $\psi = \psi' + \chi$

$\bar{\psi} = \bar{\psi}' + \bar{\chi}$

$$S = \int d^4x \, [\, \bar\psi\,(i\not\partial - m + i\varepsilon)\,\psi + \bar\eta\,\psi + \bar\psi\,\eta \,]$$

$$= \int d^4x \, [\, (\bar\psi' + \bar\chi)\,(i\not\partial - m + i\varepsilon)\,(\psi' + \chi)$$

$$+ \bar\eta\,(\psi' + \chi) + (\bar\psi' + \bar\chi)\,\eta \,]$$

$$= \int d^4x \, [\, \bar\psi'\,(i\not\partial - m + i\varepsilon)\,\psi' \qquad \sim \bar\psi' \ldots \psi'$$

$$+ \bar\chi\,(i\not\partial - m + i\varepsilon)\,\psi' + \bar\eta\,\psi' \qquad \sim \psi'$$

$$+ \bar\psi'\,(i\not\partial - m + i\varepsilon)\,\chi + \bar\psi'\,\eta \qquad \sim \bar\psi'$$

$$+ \underbrace{\bar\chi\,(i\not\partial - m + i\varepsilon)\,\chi + \bar\eta\,\chi + \bar\chi\,\eta}_{\text{terms without } \psi' \text{ or } \bar\psi'} \,]$$

*terms linear in $\psi'$ or $\bar\psi'$ should disappear*

*$\rightarrow$ appropriate choice of $\chi$ and $\bar\chi$ :*

$$(i\not\partial - m + i\varepsilon)\,\chi + \eta = 0 \;\Rightarrow\; \chi = \frac{1}{m - i\not\partial - i\varepsilon}\,\eta$$

$$\bar\chi\,(i\overleftarrow{\not\partial} - m + i\varepsilon) + \bar\eta = 0 \qquad \text{(after partial integration)}$$

$$\Rightarrow \quad S = \int d^4x \; [\, \bar{\psi}' (i\not{\partial} - m + i\varepsilon)\psi' + \bar{\eta}\,\chi \,]$$

$$= \int d^4x \; [\, \bar{\psi}' (i\not{\partial} - m + i\varepsilon)\psi' + \bar{\eta} \frac{1}{m - i\not{\partial} - i\varepsilon}\, \eta \,]$$

$$\Rightarrow \quad Z[\eta, \bar{\eta}] = \exp\left\{ \int d^4x \; \bar{\eta}(x) \frac{i}{m - i\not{\partial} - i\varepsilon}\, \eta(x) \right\}$$

$$(m - i\not{\partial}_x - i\varepsilon)\, S(x-y) = \delta^{(4)}(x-y)$$

$$\Rightarrow \quad (m - i\not{\partial}_x - i\varepsilon) \int d^4y \; S(x-y)\,\eta(y) = \eta(x)$$

$$\Rightarrow \quad \frac{1}{m - i\not{\partial}_x - i\varepsilon}\, \eta(x) = \int d^4y \;\; S(x-y)\,\eta(y)$$

$$\Rightarrow \quad Z[\eta, \bar{\eta}] = e^{\, i \int d^4x\, d^4y \; \bar{\eta}(x)\, S(x-y)\, \eta(y)}$$

$\Rightarrow$ pairing rules shown on p. 3/11