3. Dirac field

Lagrangean of (free) Dirac field:

$$\mathcal{L} = \overline{\psi}\,(i\not{\partial} - m)\,\psi = \overline{\psi}_a\, i(\gamma^\mu)_{ab}\partial_\mu \psi_b - m\,\overline{\psi}_a \psi_a$$

$\uparrow$ Dirac index ($a = 1,2,3,4$)

$\Rightarrow$ field equation $\quad (i\not{\partial} - m)\,\psi(x) = 0$

general solution of free Dirac equation:

$$\psi(x) = \sum_s \int d\mu(p)\,\Big[\, b(p,s)\, u(p,s)\, e^{-ipx} + d(p,s)^*\, v(p,s)\, e^{+ipx} \,\Big]$$

$$\not{p}\; u(p,s) = m\, u(p,s)$$

$$\not{p}\; v(p,s) = -m\, v(p,s)$$

Fourier coefficients $b, d^* \xrightarrow{\text{quantization}}$ operators $b, d^\dagger$

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = i\psi_a^\dagger = \pi_a \;, \quad \frac{\partial \mathcal{L}}{\partial \dot{\overline{\psi}}_a} = 0$$

$$\{ \psi_a(t,\vec{x}),\ \underbrace{i\psi_b^\dagger(t,\vec{y})}_{\pi_b(t,\vec{y})} \} = i\,\delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$\{A, B\} = AB + BA$ <u>anti-commutator</u>

$$\{\psi_a(x), \psi_b(y)\}\Big|_{x^0=y^0} = 0$$

$$\{\psi_a^\dagger(x), \psi_b^\dagger(y)\}\Big|_{x^0=y^0} = 0$$

equivalent to anti-commutation rules for b, d:

$$\{b(p,s), b(p',s')^\dagger\} = \{d(p,s), d(p',s')^\dagger\} =$$
$$= \delta(p,p')\,\delta_{ss'}$$

$$\{b, b\} = \{d, d\} = \{b, d\} = \{b, d^\dagger\} = 0$$

<u>energy-momentum</u> vector of the Dirac field

energy-momentum density: $T^{0\mu} = \sum_a \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} \partial^\mu \psi_a - g^{0\mu}\mathcal{L}$

$= i\psi_a^\dagger \partial^\mu \psi_a - g^{0\mu}\mathcal{L} = i\bar{\psi}\gamma^0\partial^\mu\psi - g^{0\mu}\bar{\psi}(i\not\partial - m)\psi$

second term with $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$ does not contribute because of field equation $(i\not{\partial} - m)\psi = 0$

$$\Rightarrow P^\mu = \int d^3x \; i\bar{\psi}\gamma^0\partial^\mu\psi = \int d^3x \; \psi^\dagger i\partial^\mu\psi$$

insert Fourier decomposition of ψ:

$$\overset{\text{ex.}}{\Rightarrow} P^\mu = \sum_s \int d\mu(p)\, p^\mu \left[b(p,s)^\dagger b(p,s) - d(p,s)d(p,s)^\dagger \right]$$

$$= \sum_s \int d\mu(p)\, p^\mu \left[b(p,s)^\dagger b(p,s) + d(p,s)^\dagger d(p,s) \right]$$

$$- \sum_s \underbrace{\int d\mu(p)\, p^\mu \, \delta(p,p)\,\delta_{ss}}_{\int d^3p \, p^\mu \underbrace{\delta^{(3)}(\vec{0})}_{\frac{V}{(2\pi)^3}}}$$

minus sign for fermions (in favour of super symmetry? → cancellation of bosonic contributions)

normal ordering of fermionic operators

b, d to the right of all $b^\dagger, d^\dagger$; factor (-1) if two fermionic operators ar interchanged

$$\Rightarrow \quad P^\mu = \int d^3x \, : \psi^\dagger i \partial^\mu \psi : =$$

$$= \sum_s \int d\mu(p) \, p^\mu \, [b(p,s)^\dagger b(p,s) + d(p,s)^\dagger d(p,s)]$$

$$= \sum_s \int [dn(p,s) + d\bar{n}(p,s)] \, p^\mu$$

$$dn(p,s) = d\mu(p) \, b(p,s)^\dagger b(p,s)$$

$$d\bar{n}(p,s) = d\mu(p) \, d(p,s)^\dagger d(p,s)$$

$$[P^\mu, b(p,s)] = -p^\mu b(p,s)$$

$$[P^\mu, b(p,s)^\dagger] = p^\mu b(p,s)^\dagger$$

$$[P^\mu, d(p,s)] = -p^\mu d(p,s)$$

$$[P^\mu, d(p,s)^\dagger] = p^\mu d(p,s)^\dagger$$

$$\left.\begin{array}{l} b(p,s) \\ d(p,s) \end{array}\right. \underline{\text{annihilates}} \text{ a } \begin{array}{c} \text{particle} \\ \text{antiparticle} \end{array} \text{ with momentum } p^\mu \text{ and spin } s$$

$$\left.\begin{array}{l} b(p,s)^\dagger \\ d(p,s)^\dagger \end{array}\right. \underline{\text{creates}} \text{ a } \begin{array}{c} \text{particle} \\ \text{antiparticle} \end{array} \text{ with momentum } p^\mu \text{ and spin } s$$

why <u>anti</u>particle ?

$\bar\psi \gamma^\mu \psi$ is a conserved current

$$j^\mu = q\, \bar\psi \gamma^\mu \psi$$

$$Q = \int d^3x \, :j^0(x): \; = q \int d^3x \, :\psi^\dagger(x)\, \psi(x):$$

$$\overset{\text{ex.}}{=} q \sum_s \int \left[dn(p,s) - d\bar n(p,s) \right]$$

$$\Rightarrow \quad [Q, b(p,s)] = -q\, b(p,s)$$

$$[Q, b(p,s)^\dagger] = q\, b(p,s)^\dagger$$

$$[Q, d(p,s)] = q\, d(p,s)$$

$$[Q, d(p,s)^\dagger] = -q\, d(p,s)^\dagger$$

i.e. $b(p,s)^\dagger$ creates a state with charge $+q$

$d(p,s)^\dagger$ — " — — " — — " — — " — — " — $-q$

construction of <u>Fock space</u>:

vacuum state defined by

$$b(p,s)\,|0\rangle = d(p,s)\,|0\rangle = 0 \quad \forall \vec{p}, s$$

further states obtained by applying products of $b^\dagger$ and $d^\dagger$ on $|0\rangle$

Pauli exclusion principle guaranteed by anti-commutation relations

e.g. $$b(p,s)^\dagger b(p,s)^\dagger |0\rangle =$$

$$= - b(p,s)^\dagger b(p,s)^\dagger |0\rangle$$

$$\Rightarrow \quad b(p,s)^\dagger b(p,s)^\dagger |0\rangle = 0$$

two identical fermions cannot sit in the same one-particle state

Feynman propagator of the Dirac field

$$\langle 0|\psi_a(x)\,\psi_b(y)|0\rangle = 0$$

$$\left(\langle 0|(b+d^\dagger)(b+d^\dagger)|0\rangle = \langle 0|b\,d^\dagger|0\rangle = \right.$$

$$\left. = -\langle 0|d^\dagger b|0\rangle = 0\right)$$

$$\langle 0|\psi_a(x)\,\bar{\psi}_b(y)|0\rangle \qquad (\langle 0|(b+\cancel{d^\dagger})(b^\dagger+\cancel{d})|0\rangle$$

$$= \sum_{s,s'}\int d\mu(p)\,d\mu(p')\;u_a(p,s)\,e^{-ipx}\,\bar{u}_b(p',s')\,e^{+ip'y}$$

$$\underbrace{\langle 0|\,b(p,s)\;b(p',s')^\dagger|0\rangle}_{\delta_{ss'}\,\delta(p,p')}$$

$$= \int d\mu(p)\,\underbrace{\sum_s u_a(p,s)\,\bar{u}_b(p,s)}_{(\not{p}+m)_{ab}}\,e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{ab}\int d\mu(p)\,e^{-ip(x-y)}$$

$$\langle 0 | \overline{\psi}_b(y)\, \psi_a(x) | 0 \rangle \qquad (\langle 0 | (\cancel{b^\dagger} + d)(\cancel{b} + d^\dagger) | 0 \rangle$$

$$= \sum_{s,s'} \int d\mu(p)\, d\mu(p')\ \overline{v}_b(p',s')\, e^{-ip'y}\, v_a(p,s)\, e^{+ipx}$$

$$\underbrace{\langle 0 | d(p',s')\, d(p,s)^\dagger | 0 \rangle}_{\delta_{ss'}\, \delta(p,p')}$$

$$= \int d\mu(p)\ \underbrace{\sum_s v_a(p,s)\, \overline{v}_b(p,s)}_{(\not{p} - m)_{ab}}\ e^{ip(x-y)}$$

$$= -\,(i\not{\partial}_x + m)_{ab} \int d\mu(p)\, e^{ip(x-y)}$$

definition of <u>time ordering</u> for fermionic fields:

$$T\, \psi_a(x)\, \overline{\psi}_b(y) := \begin{cases} \psi_a(x)\, \overline{\psi}_b(y) & \text{for } x^0 > y^0 \\ -\overline{\psi}_b(y)\, \psi_a(x) & \text{for } x^0 < y^0 \end{cases}$$

$$\Rightarrow \langle 0 | T \psi_a(x)\, \overline{\psi}_b(y) | 0 \rangle =$$

$$= \Theta(x^0 - y^0)\, \langle 0 | \psi_a(x)\, \overline{\psi}_b(y) | 0 \rangle$$

$$- \Theta(y^0 - x^0)\, \langle 0 | \overline{\psi}_b(y)\, \psi_a(x) | 0 \rangle$$

$$= \Theta(x^0-y^0)\,(i\not\partial_x + m)_{ab} \int d\mu(p)\, e^{-ip(x-y)}$$

$$+ \Theta(y^0-x^0)\,(i\not\partial_x + m)_{ab} \int d\mu(p)\, e^{+ip(x-y)}$$

$$= (i\not\partial_x + m)_{ab} \underbrace{\left[\Theta(x^0-y^0)\int d\mu(p)\, e^{-ip(x-y)} + \Theta(y^0-x^0)\int d\mu(p)\, e^{+ip(x-y)}\right]}_{\frac{1}{i}\Delta(x-y)}$$

explonation of lost skp:

$$\left[\frac{\partial}{\partial x^0}\Theta(x^0-y^0)\right]\int d\mu(p)\, e^{-ipx} + \left[\frac{\partial}{\partial x^0}\Theta(y^0-x^0)\right]\int d\mu(p)\, e^{+ip(x-y)}$$

$$= \delta(x^0-y^0)\left[\int d\mu(p)\, e^{-ip(x-y)} - \int d\mu(p)\, e^{+ip(x-y)}\right] = 0$$

$$\Rightarrow \langle 0 | T\, \psi_a(x)\, \overline{\psi}_b(y) | 0 \rangle = \frac{1}{i}\,(i\not\partial_x + m)_{ab}\,\Delta(x-y)$$

$$=: \frac{1}{i}\, S(x-y)$$

$$\Rightarrow (m - i\not\partial_x)\, S(x-y) =$$

$$= (m - i\not\partial_x)(m + i\not\partial_x)\,\Delta(x-y)$$

$$= (m^2 + \Box_x)\,\Delta(x-y) = \delta^{(4)}(x-y)$$

$\Rightarrow$ $S(x)$ is a Green's function of $m - i\not{\partial}$ with Feynman boundary conditions

Fourier representation of $S(x-y)$ can be obtained from the Fourier representation of the scalar Green's function $\Delta(x-y)$:

$$S(x-y) = (i\not{\partial}_x + m)\, \Delta(x-y)$$

$$= (i\not{\partial}_x + m) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m^2 - k^2 - i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \frac{\not{k} + m}{m^2 - k^2 - i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m - \not{k} - i\varepsilon}$$

$$\langle 0 | T \, \psi_a(x) \, \psi_b(y) | 0 \rangle = 0$$

$$\langle 0 | T \, \overline{\psi}_a(x) \, \overline{\psi}_b(y) | 0 \rangle = 0$$

vacuum expectation value of time-ordered product of arbitrary number of Dirac field operators:

<u>nonvanishing</u> result only if # of ψs = # of $\overline{\psi}$s:

$$\langle 0 | T \, \psi_{a_1}(x_1) \, \overline{\psi}_{b_1}(y_1) \ldots \psi_{a_n}(x_n) \, \overline{\psi}_{b_n}(y_n) | 0 \rangle$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma \, \frac{1}{i} \, S_{a_1 b_{\sigma(1)}}(x_1 - y_{\sigma(1)}) \ldots \frac{1}{i} \, S_{a_n b_{\sigma(n)}}(x_n - y_{\sigma(n)})$$

$$(-1)^\sigma = \begin{cases} +1 & \text{if } \sigma \text{ even permutation of } 1, \ldots, n \\ -1 & \text{—''— —''— odd —''— —''— —''—} \end{cases}$$