1. Scalar field (spin 0)

real scalar field \[ \phi(x) = \phi(x)^* \]

\[ \phi'(x') = \phi(x) \quad x' = L x + \alpha \quad (L \in \mathbb{L}_+) \]

free scalar field \rightarrow field equation \((\Box + m^2)\phi(x) = 0\)

\((Klein-Gordon\ equation)\)

action integral \[ S = \int d^4x \left\{ \frac{1}{2} \left( \partial\mu \partial^\mu \phi - m^2 \phi^2 \right) \right\} \quad \mathcal{L} \]

\(\text{Lagrangian density}\)

equation of motion \[ \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \]

\[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi \quad \frac{\partial \mathcal{L}}{\partial \phi} = - m^2 \phi \Rightarrow (\Box + m^2)\phi = 0 \]

Lagrangian \rightarrow Hamiltonian

\[ \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \quad \text{canonical momentum conjugate to } \phi \]
Hamilton density \[ H = \pi \dot{\phi} - L = \frac{1}{2} \left[ \pi^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2 \right] \]

no explicit \( x \)-dependence of \( L \) \( \rightarrow \) energy-momentum conservation (inv. under space-time translations)

energy momentum tensor \[ T_{\mu \nu} = \frac{\partial L}{\partial \phi_{,\mu}} \phi_{,\nu} - g_{\mu \nu} L \]

\[ \partial \tau \ T_{\mu \nu} = 0 \]

\[ \Rightarrow \quad \mathcal{P}^\mu = \int d^3 x \ T^{0 \mu}(x) = \text{const.} \quad \text{4-momentum} \]

\[ \mathcal{P} = - \int d^3 x \ \pi \ \vec{\nabla} \phi \quad \text{3-momentum of scalar field} \]

invariance under rotations \( \rightarrow \) angular momentum cons.

\[ \mathcal{L} = - \int d^3 x \ \pi \ \vec{x} \times \vec{\nabla} \phi \]

quantization (canonical quantization)

equal time commutation relations

\[ [\phi(t, \vec{x}), \pi(t, \vec{y})] = i \delta^{(3)}(\vec{x} - \vec{y}) \]

\[ [\phi(t, \vec{x}), \phi(t, \vec{y})] = [\pi(t, \vec{x}), \pi(t, \vec{y})] = 0 \]
Fourier decomposition

\[ \phi(x) = \int d^3 \mu(p) \left[ a(p) e^{-i p \cdot x} + a(p)^* e^{i p \cdot x} \right] \]

\[ p^0 = \sqrt{m^2 + \vec{p}^2} = \omega(p) \]

\[ p \cdot x = p^0 t - \vec{p} \cdot \vec{x} \]

\[ a(p) = i \int d^3 x e^{i p \cdot x} \frac{\partial}{\partial x} \phi(x) \quad \text{(exercise)} \]

\[ \vec{A} \cdot \vec{B} := A \cdot B - (\vec{a}B) \]

\[ [a(p), a(p')^+] = (2 \pi)^3 2 p^0 \frac{\delta^{(3)}(\vec{p} - \vec{p}')}{\delta(p, p')} \]

\[ [a(p), a(p')^+] = [a(p)^+, a(p')^+] = 0 \]

\[ H' = \int d^3 x \ H = \frac{1}{2} \int d^3 \mu(p) p^0 \left\{ a(p)^+ a(p) + a(p) a(p)^+ \right\} \]

\[ = \int d^3 \mu(p) p^0 a(p)^+ a(p) + \frac{1}{2} \int d^3 \mu(p) p^0 \delta(p, p) \quad \text{Evac} \]
vacuum energy  \( E_{\text{vac}} = \frac{1}{2} \int \! \! \int \! \! \int d^3p \ p^0 \ S^{(3)}(\vec{r}) \)

\[
S^{(3)}(\vec{r}) = \lim_{\vec{p}' \to \vec{p}} S^{(3)}(\vec{p}' - \vec{p}) = \lim_{\vec{p}' \to \vec{p}} \int d^3x \ e^{i(\vec{p}' - \vec{p}) \cdot \vec{x}} \frac{1}{(2\pi)^3}
\]

\[
\to \frac{V}{(2\pi)^3} \quad \text{in finite volume } V \quad \text{(IR divergence)}
\]

energy density  \( \varepsilon_{\text{vac}} = E_{\text{vac}} / V = \frac{1}{2 \ (2\pi)^3} \int \! \! \int \! \! \int d^3p \ p^0 \)

but even energy density  \( \text{UV divergent}! \)

\[
\varepsilon_{\text{vac}} = \frac{1}{2 \ (2\pi)^3} \int \! \! \int \! \! \int d^3p \ \sqrt{\vec{p}^2 + m^2} \uparrow \frac{4\pi}{2 \ (2\pi)^3} \int \! \! \int d^0 p \ \sqrt{p^2 + m^2} \quad \text{UV cut-off}
\]

\[
\sim \frac{1}{(2\pi)^2} \frac{\Lambda^4}{4} \quad \Lambda \to \infty \quad \to \infty
\]

vacuum energy  \( E_{\text{vac}} \) can be removed by "renormalization"

\[
H' \to H = \int \! \! \int d^4p (\hat{p} \ p^0 \ a(p)^\dagger \ a(p))
\]
can be formally achieved by normal ordering $\mathcal{H}$ of the energy density: rearrange the order of the factors such that all creation operators stand to the left of all annihilation operators

\[
\text{Field momentum: } \quad \vec{P} = -\int d^3 \pi \vec{\nabla} \phi = \int d\mu(p) \vec{p} a(\vec{p})^\dagger a(\vec{p}) + \vec{P}_{\text{vac}}
\]

where $\vec{P}_{\text{vac}} = \frac{i}{2} \int d\vec{p} \vec{p} S^{(G)}(\vec{0})$

0 for rotation invariant regularization

$\vec{P}_{\text{vac}}$ automatically removed by normal ordering:

\[
\vec{P} = -\int d^3 \pi \vec{\nabla} \phi = \int d\mu(p) \vec{p} a(p)^\dagger a(p) \vec{p} \\
\Rightarrow P^\mu = \int d\mu(p) \vec{p} a(p)^\dagger a(p) p^\mu \quad 4\text{-momentum}
\]

$\Rightarrow [P^\mu, a(p)] = -p^\mu a(p)$

$\Rightarrow [P^\mu, a(p)^\dagger] = p^\mu a(p)^\dagger$ (exercises)
\[
\exp(iPa) \phi(x) \exp(-iPa) = \phi(x+a)
\]

\(P^t\) generates space-time translations

Ground state (vacuum state) |0\rangle characterized by

\[a(p)|0\rangle = 0 \quad \forall p\]

\[\Rightarrow \quad P^t|0\rangle = 0\]

One-particle momentum eigenstates: |p\rangle = a(p)^\dagger |0\rangle

\[P^t|p\rangle = P^t a(p)^\dagger |0\rangle = p^t|p\rangle\]

\[\quad a(p)^\dagger P^t + p^t a(p)^\dagger\]

Normalization \[\langle p'|p\rangle = (2\pi)^3 2p^0 \delta^3(p' - p)\]

\[N = \int d\mu(p) a(p)^\dagger a(p) \quad \text{particle number operator}\]
\begin{align*}
\text{n-particle energy-momentum eigenstates:} & \\
|p_1, p_2, \ldots, p_n\rangle &= \hat{a}(p_1)^\dagger \hat{a}(p_2)^\dagger \ldots \hat{a}(p_n)^\dagger |0\rangle \\
\mathcal{P}^{n} |p_1, \ldots, p_n\rangle &= (p_1^{\alpha} + \ldots + p_n^{\alpha}) |p_1, \ldots, p_n\rangle \\
\langle p_1, \ldots, p_n | R_1, \ldots, R_n \rangle &= \sum_{\sigma \in S_n} \prod_{i=1}^{n} \delta(p_i, R_{\sigma(i)}) \\
&\uparrow \text{permutations of } n \text{ elements} \\
\text{projection operator on subspace of n-particle states} & \\
\mathcal{P}^{\text{ch}} = \frac{1}{n!} \int d\mu(p_1) \ldots d\mu(p_n) |p_1, \ldots, p_n\rangle \langle p_1, \ldots, p_n| \\
\text{normalized n-particle state:} & \\
|\psi^{(n)}\rangle &= \frac{1}{n!} \int d\mu(p_1) \ldots d\mu(p_n) |p_1, \ldots, p_n\rangle \langle p_1, \ldots, p_n| \psi^{(n)}\rangle \\
&\underbrace{\psi^{(n)}(p_1, \ldots, p_n)}_{\text{totally symmetric wave function}} \\
&\text{(Bose statistics)}
\end{align*}
\[ \langle \psi^{(n)} | \psi^{(n)} \rangle = 1 \iff \frac{1}{n!} \int d\mu(p_1) \cdots d\mu(p_n) \left| \psi^{(n)}(p_1, \ldots, p_n) \right|^2 = 1 \]

**Fock space**

\[ \mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \cdots \]

spanned by \[ |0\rangle \quad |p\rangle \quad |p_1, p_2\rangle \cdots \]

\[ |\rangle = \sum_{n=0}^{\infty} \mathcal{P}^{(n)} \quad , \quad \mathcal{P}^{(0)} = |0\rangle \langle 0| \]

**n-point functions** (Green's functions, correlation functions)

vacuum expectation values of time-ordered products of field operators

\[ \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle \]

\[ T \phi(x_1) \cdots \phi(x_n) = \phi(x_{i_1}) \cdots \phi(x_{i_n}) \]

\[ i_{11} \ldots , \text{in permutation of } 1, \ldots, n \text{ such that} \]

\[ x_{i_1}^0 > x_{i_2}^0 > \cdots > x_{i_n}^0 \]
**interacting theory** $\rightarrow$ **S-matrix elements** can be extracted from **n-point functions** (LSZ)

**free theory** $\rightarrow$ **n-point functions** can be written as sum of products of **two-point functions**

$\rightarrow$ consider $\langle 0 | T \phi(x) \phi(y) | 0 \rangle$

for **free** scalar field

$\rightarrow$ plays central role in **perturbation expansion** of **interacting theories**

**theory translation invariant** $\rightarrow$ **2-point function** depends only on difference $x-y$:

$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \langle 0 | T \phi(x-y) \phi(0) | 0 \rangle$

**free propagator**

$\Delta(x) := i \langle 0 | T \phi(x) \phi(0) | 0 \rangle$
Remark: \( \Delta(-x) = \Delta(x) \)

\[
\langle 0 | T \Phi(x) \Phi(0) | 0 \rangle = \Theta(x^0) \langle 0 | \Phi(x) \Phi(0) | 0 \rangle + \Theta(-x^0) \langle 0 | \Phi(0) \Phi(x) | 0 \rangle
\]

\[
\langle 0 | \Phi(x) \Phi(0) | 0 \rangle =
\]

\[
= \langle 0 | \int d\mu(p) \left[ e^{-ipx} a(p) + e^{ipx} a(p)^\dagger \right] \int d\mu(R) \left[ a(R) + a(R)^\dagger \right] | 0 \rangle
\]

\[
= \int d\mu(p) d\mu(R) e^{-ipx} \langle 0 | a(p) a(R)^\dagger | 0 \rangle \underbrace{[a(p), a(R)^\dagger]}_{[a(p), a(R)^\dagger] = \delta(p, R)}
\]

\[
= \int d\mu(p) e^{-ipx}
\]

analogously:
\[
\langle 0 | \Phi(0) \Phi(x) | 0 \rangle = \int d\mu(p) e^{ipx}
\]
\[ \Delta (x) = i \Theta(x^0) \int dp^4 \frac{e^{-ipx}}{m^2 - p^2 - i\varepsilon} + i \Theta(-x^0) \int dp^4 \frac{e^{ipx}}{m^2 - p^2 - i\varepsilon} \]

only "positive" frequencies for \( x^0 > 0 \)

only "negative" frequencies for \( x^0 < 0 \)

\[ (\Box + m^2) \Delta (x) = \delta^{(4)}(x) \]

\( \Delta(x) \) is Green's function of Klein-Gordon equation with the following boundary conditions: only pos. frequ. for \( x^0 > 0 \) and only neg. frequ. for \( x^0 < 0 \) (Feynman boundary conditions)

\( \Delta(x) \) can be written as Fourier integral

\[ \Delta(x) = \int \frac{dp^4}{(2\pi)^4} \frac{e^{-ipx}}{m^2 - p^2 - i\varepsilon} \]

Feynman boundary conditions taken into account by \( m^2 \rightarrow m^2 - i\varepsilon \); in this formula, \( p^0 \) is an integration variable and not \( \sqrt{p^2 + m^2} \)!
complex scalar field

\[ \phi(x) = \frac{1}{i}\left[ \phi_1(x) + i \phi_2(x) \right] \quad , \quad \phi_i^* = \phi_i \]

\[ (\Box + m^2) \phi(x) = 0 \quad \Leftrightarrow \quad (\Box + m^2) \phi_i(x) = 0 \]

\[ \mathcal{L} = \frac{i}{2} \sum_{i=1}^{2} \left( \partial_\mu \phi_i \partial^\mu \phi_i^* - m^2 \phi_i \phi_i^* \right) = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \]

conserved current \quad \mathcal{j}_\mu = i \phi^* \partial^\mu \phi

reason: \quad \mathcal{L} \text{ invariant under (global) } U(1) \text{ gauge transformation } \phi(x) \to e^{i \alpha} \phi(x) , \quad \alpha \in \mathbb{R}

(exercises)

canonical quantization:

\[ [a_i(p), a_j(p')^\dagger] = \delta_{ij} \delta(p, p') \]

(all other commutators vanish)

\[ \phi_i(x) = \int dp(p) \left[ a_i(p) e^{-ipx} + a_i^\dagger(p) e^{+ipx} \right] \]
\[ \Rightarrow \phi(x) = \int \! dp \, \left[ a_+(p) e^{-ipx} + a_-(p) e^{ipx} \right] \]

\[ a_+(p) = \frac{1}{\sqrt{2}} \left[ a_1(p) + i a_2(p) \right] \]

\[ a_-(p) = \frac{1}{\sqrt{2}} \left[ a_1(p) + i a_2(p) \right] \]

\[ \left[ a_+(p), a_+(p')^\dagger \right] = \left[ a_-(p), a_-(p')^\dagger \right] = \delta(p,p') \]

(all other commutators vanish)

\[ \Rightarrow \left[ \phi(x), \phi^+(y) \right] \bigg|_{x^0 = y^0} = i \delta^{(3)}(x-y) \]

consistent with \[ \frac{\partial L}{\partial \dot{\phi}(x)} = \pi(x) = \phi^+(x) \]

\[ i \langle 0 \vert T \phi(x) \phi(y)^+ \vert 0 \rangle = \Delta(x-y) \quad \text{exercise} \]

particle number operator

\[ N = \sum_{i=1}^{2} \left( \int \! dp \, \left[ a_i(p)^\dagger a_i(p) \right] \right) \]

\[ = \int \! dp \, a_+(p)^\dagger a_+(p) + \int \! dp \, a_-(p)^\dagger a_-(p) \]

\[ = \int \! dp \, \frac{a_+(p)^\dagger a_+(p)}{dn_+(p)} + \int \! dp \, \frac{a_-(p)^\dagger a_-(p)}{dn_-(p)} \]
charge operator

\[ Q = \int d^3x \cdot j^c(x) \cdot = \int d\mu(p) \left[ a_+(p)^\dagger a_+(p) - a_-(p)^\dagger a_-(p) \right] \]

\[ = \int \left[ dn_+(p) - dn_-(p) \right] \]  \hspace{1cm} (exercise)

energy-momentum operator

\[ P^\mu = \sum_{i=1}^{2} \int d\mu(p) \ a_i(p)^\dagger a_i(p) \ p^\mu \]

\[ = \int d\mu(p) \left[ a_+(p)^\dagger a_+(p) + a_-(p)^\dagger a_-(p) \right] p^\mu \]

\[
\left\{ \begin{align*}
[P^\mu, Q] &= 0 \\
[P^\mu, P^\nu] &= 0
\end{align*} \right. \Rightarrow \exists \ \text{ONB of eigenvectors of } P^\mu, Q
\]

\[
[Q, a_\pm(p)^\dagger] = \pm a_\pm(p)^\dagger
\]

vacuum \( |0\rangle \): \( a_\pm(p) |0\rangle \ \forall \ p \)

\[ \Rightarrow Q |0\rangle = 0 \]
\[ Q \, a_\pm (p)^+ \, 10 > = \pm \, a_\pm (p)^+ \, 10 > \]

\[ \Rightarrow \, |p, \pm > : = a_\pm (p)^+ \, 10 > \, \text{eigenstates of} \, Q \text{ with eigenvalues} \pm 1 \]

\[ a_\pm (p)^+ \text{ creates state with charge} \pm 1 \]

\[ a_\pm (p) \text{ destroys} \]

non-hermitean scalar field describes particle and the associated antiparticle

examples: \( \pi^\pm \) (\( Q = \text{electromagnetic charge} \) in units of elementary charge \( e \))

\( K^0 \bar{K}^0 \) (\( "\text{charge}" = \text{strangeness} \, S = \pm 1 \))

charge conjugation

interchange particle \( \leftrightarrow \) antiparticle
field operator

\[ \Phi(x) = \int dp(p) \left[ a_+(p) e^{-i p x} + a_-(p)^\dagger e^{i p x} \right] \]

change conjugate field:

\[ \Phi^c(x) = \int dp(p) \left[ a_-(p) e^{-i p x} + a_+(p)^\dagger e^{i p x} \right] \]

\( \mathcal{L} \) invariant under \( \Phi \to \Phi^* \) (discrete symmetry)

\exists \text{ unitary operator } \mathcal{C} \quad (\mathcal{C}^\dagger \mathcal{C} = \mathcal{C} \mathcal{C}^\dagger = 1)

with \( \mathcal{C} \Phi(x) \mathcal{C}^{-1} = \Phi(x)^\dagger \Rightarrow \mathcal{C} j^\mu \mathcal{C}^{-1} = -j^\mu \)

\[ \mathcal{C} Q \mathcal{C}^{-1} = -Q \]

\[ \mathcal{C} a_+(p) \mathcal{C}^{-1} = a_-^\dagger(p) \Rightarrow \mathcal{C} a_+(p)^\dagger \mathcal{C}^{-1} = a_-^\dagger(p)^\dagger \]

\[ \Rightarrow \mathcal{C} |0\rangle = |0\rangle \quad \text{(phase can be absorbed in } |0\rangle \text{)} \]

\[ \mathcal{C} |p, \pm\rangle = \mathcal{C} a_+(p)^\dagger |0\rangle = a_-^\dagger(p) \mathcal{C} |0\rangle = a_-^\dagger |0\rangle = |p, \mp\rangle \]