Exercises "Particle Physics II" (2010)

20. Verify the commutation relations of P^μ and Q with the creation and annihilation operators listed in the lecture notes.

21. Show:
$$i\,[P^\mu, \psi(x)] = \partial^\mu \psi(x) \ .$$
Discuss the space-time shift
$$\exp(ia \cdot P)\, \psi(x)\, \exp(-ia \cdot P)$$
following from this relation.

22. The γ matrices are 4×4 matrices satisfying the anticommutation relations
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4 \ .$$
Show:
    (a) $\not{a}\not{a} = a \cdot a\, \mathbb{1}_4$ where $\not{a} := a_\mu \gamma^\mu$
    (b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
    (c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$
    (d) The trace of the product of an odd number of γ matrices vanishes. Hint: $\gamma_5^2 = \mathbb{1}_4$, $\{\gamma^\mu, \gamma_5\} = 0$

23. Show the following relations:
$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m \ , \quad \sum_s v(p,s)\bar{v}(p,s) = \not{p} - m \ .$$
Hint: Consult the lecture notes of "Particle Physics I".

24. The generating functional of the free Dirac field was found to be
$$\begin{aligned} Z[\eta,\bar{\eta}] &\equiv \left\langle 0 \left| T \exp\left\{ i \int d^4x\ \left[\bar{\eta}(x)\Psi(x) + \overline{\Psi}(x)\eta(x)\right] \right\} \right| 0 \right\rangle \\ &= \exp\left\{ i \int d^4x\, d^4y\, \bar{\eta}(x) S(x-y) \eta(y) \right\} \ . \end{aligned}$$
Using this formula, compute
$$\langle 0 | T\{\Psi_{a_1}(x_1)\overline{\Psi}_{b_1}(y_1) \ldots \Psi_{a_n}(x_n)\overline{\Psi}_{b_n}(y_n)\} | 0 \rangle \ .$$

25. Derive the field equation of a free real vector field V_μ with mass $M \neq 0$ from the Lagrangean

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2}{2} V_\mu V^\mu, \quad V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu.$$

26. Discuss the plane wave solutions $\varepsilon^\mu e^{\pm ikx}$ of the free vector field.

 (a) Show that $p^\mu = (M, \vec{0})$ implies $\varepsilon^\mu = (0, \vec{\varepsilon})$.

 (b) Choose the three pairwise orthogonal unit vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

 for the three linear independent polarizations of $\vec{\varepsilon}$. Perform a Lorentz boost L^μ_ν in the direction of $\vec{e}_3$ such that

$$Lp = k = \begin{pmatrix} \sqrt{M^2 + k_z^2} \\ 0 \\ 0 \\ k_z \end{pmatrix}.$$

 (c) Determine

$$\varepsilon(k,1) = L \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon(k,2) = L \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varepsilon(k,3) = L \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

 (d) Verify the relations

$$\varepsilon^\mu(k,\lambda)\varepsilon_\mu(k,\lambda') = -\delta_{\lambda\lambda'}, \quad \sum_{\lambda=1}^{3} \varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda') = -g^{\mu\nu} + k^\mu k^\nu / M^2.$$

27. Find the Green's function of the differential operator

$$g_{\mu\nu}(\Box - i\varepsilon) - (1 - \xi)\partial_\mu \partial_\nu$$

(photon propagator).