Exercises "Particle Physics II" (2010)

12. The one-dimensional harmonic oscillator is described by the Hamilton operator

$$H = \frac{P(t)^2}{2m} + \frac{m\omega^2 Q(t)^2}{2} \ .$$

The position operator $Q(t)$ and the momentum operator $P(t)$ fulfil the canonical commutation relation

$$[Q(t), P(t)] = i\hbar \mathbb{1} \ .$$

   (a) Verify that Heisenberg's equation of motion for $Q(t)$,

$$\dot{Q}(t) = \frac{i}{\hbar}\left[H, Q(t)\right] \ ,$$

   implies the classical equation of motion $\ddot{Q}(t) + \omega^2 Q(t) = 0$.

   (b) Express $Q(t)$ in terms of the ladder operators a and $a^\dagger$ which satisfy the commutation relation $\left[a, a^\dagger\right] = \mathbb{1}$.

   (c) Calculate the two-point function $\langle 0|TQ(t_1)Q(t_2)|0\rangle$.

13. Determine the generating functional of the one-dimensional harmonic oscillator,

$$Z[f] = \langle 0|Te^{\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty} dt\, f(t)Q(t)}|0\rangle \ ,$$

using the path integral representation

$$Z[f] = \frac{1}{\mathcal{N}}\int [dq] \exp\left\{\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty} dt \left[\frac{m}{2}\dot{q}(t)^2 - \frac{m(\omega^2 - i\varepsilon)}{2}q(t)^2 + f(t)q(t)\right]\right\} \ .$$

Give a physical interpretation of the external field $f(t)$. Verify the result for the two-point function obtained with the operator method.

Hint: The path integral calculation is completely analogous to the one for a free field, discussed in detail in the lecture. The position variable $q(t)$ can be interpreted as a scalar field living in $0+1$-dimensional spacetime.

14. The generating functional of a free **non-hermitean** scalar field $\phi(x)$,

$$Z[f] = \langle 0|Te^{i\int d^4x\left(f(x)^*\phi(x) + f(x)\phi(x)^\dagger\right)}|0\rangle \ ,$$

can be deduced from the generating functionals of two hermitean scalar fields $\phi_{1,2}(x)$ with equal masses. Use this relation to derive the explicit form of $Z[f]$.

Hint: $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $f = (f_1 + if_2)/\sqrt{2}$, $f_i^* = f_i$.

15. Using the result of the previous problem, discuss the pairing rule for the Green's function

$$\langle 0|T\phi(x_1)\dots\phi(x_m)\phi(y_1)^\dagger\dots\phi(y_n)^\dagger|0\rangle$$

of a non-hermitean scalar field $\phi(x)$.

16. Use Noether's theorem to derive the form of the conserved current j^μ associated with the global $U(1)$ symmetry

$$\psi(x)\to e^{i\alpha}\psi(x)\ ,\quad \bar\psi(x)\to e^{-i\alpha}\bar\psi(x)$$

of the Lagrangean $\mathcal{L}=\bar\psi(i\partial\!\!\!/-m)\psi$.

17. Use the Dirac equation to show that $\partial_\mu j^\mu=0$ is indeed fulfilled for

$$j^\mu = q\bar\psi\gamma^\mu\psi\ .$$

18. The Fourier decomposition of the field operator of a free Dirac field is given by

$$\psi(x)=\sum_s\int d\mu(p)\ \left[b(p,s)u(p,s)e^{-ip\cdot x}+d(p,s)^\dagger v(p,s)e^{ip\cdot x}\right]\ .$$

The creation and annihilation operators fulfil the anticommutation relations

$$\begin{aligned}\left\{b(p,s),\,b(p',s')^\dagger\right\} &= \left\{d(p,s),\,d(p',s')^\dagger\right\}=\delta(p,p')\delta_{ss'}\ ,\\ \left\{b(p,s),\,b(p',s')\right\} &= \left\{d(p,s),\,d(p',s')\right\}=0\ ,\\ \left\{b(p,s),\,d(p',s')\right\} &= \left\{b(p,s),\,d(p',s')^\dagger\right\}=0\ .\end{aligned}$$

Using these relations, verify the canonical anticommutaion relations for the field operator:

$$\left\{\psi_a(t,\vec x),\,\psi_b(t,\vec y)^\dagger\right\}=\delta_{ab}\delta^{(3)}(\vec x-\vec y)\ ,\quad \left\{\psi_a(x),\,\psi_b(y)\right\}\Big|_{x^0=y^0}=0\ .$$

19. Express the energy-momentum operator

$$P^\mu=\int d^3x\ :\psi^\dagger i\partial^\mu\psi:$$

and the charge operator

$$Q=q\int d^3x\ :\psi^\dagger\psi:$$

in terms of creation and annihilation operators.