Exercises "Particle Physics II" (2010)

1. The commutation relations of the creation and annihilation operators of a Hermitean scalar field (spin 0 field) with mass m are given by

$$\left[a(p), a(p')^\dagger\right] = \underbrace{(2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}\,')}_{\delta(p,p')} \ , \quad [a(p), a(p')] = 0 \ ,$$

where $p^0 = \sqrt{m^2 + \vec{p}^{\,2}}$. The one-particle momentum eigenstate $|p\rangle$ is defined by $|p\rangle = a(p)^\dagger|0\rangle$. The general form of a normalizable one-particle state $|\psi^{(1)}\rangle$ is given by

$$|\psi^{(1)}\rangle = \int \underbrace{\frac{d^3p}{(2\pi)^3 2p^0}}_{d\mu(p)} |p\rangle\, \psi^{(1)}(p) \ .$$

   (a) $\langle p|p'\rangle =?$

   (b) Determine the normalization condition for the momentum-space wave function $\psi^{(1)}(p)$ implied by the state normalization $\langle\psi^{(1)}|\psi^{(1)}\rangle = 1$.

   (c) Show that the projection operator

$$P^{(1)} = \int d\mu(p)|p\rangle\langle p|$$

   satisfies indeed $P^{(1)}P^{(1)} = P^{(1)}$.

2. The two-particle momentum eigenstate $|p_1, p_2\rangle$ is defined by

$$|p_1, p_2\rangle = a(p_1)^\dagger a(p_2)^\dagger|0\rangle \ .$$

   (a) $\langle p_1, p_2|p_1' p_2'\rangle =?$

   (b) Determine the operator $P^{(2)}$ projecting on the two-particle subspace.

   (c) Discuss the general form of a normalizable two-particle state $|\psi^{(2)}\rangle$ and the properties of the corresponding two-particle wave function in momentum space.

3. In the case of n particles, one defines

$$|p_1, \ldots p_n\rangle = a(p_1)^\dagger \ldots a(p_n)^\dagger|0\rangle \ .$$

   Show by induction:

$$\langle p_1, \ldots p_n|k_1, \ldots k_n\rangle = \sum_{\sigma\in\mathcal{S}_n} \prod_{i=1}^{n} \delta(p_i, k_{\sigma(i)}) \ .$$

4. Discuss the projection operator $P^{(n)}$ and the general form of an n-particle state $\psi^{(n)}$.

5. The Fourier decomposition of a real scalar field is given by

$$\phi(x) = \int d\mu(p) \left[a(p)e^{-ipx} + a(p)^\dagger e^{ipx}\right] .$$

Show that $a(p)$ and $a(p)^\dagger$ can be obtained from $\phi(x)$ by the relations

$$a(p) = i \int d^3x\, e^{ipx} \overleftrightarrow{\partial_0}\, \phi(x) , \quad a(p)^\dagger = -i \int d^3x\, e^{-ipx} \overleftrightarrow{\partial_0}\, \phi(x) .$$

6. Use the previous formulas to show that the canonical equal-time commutation relations for ϕ and $\dot{\phi}$ imply the commutation relations for $a(p)$ and $a(p)^\dagger$ displayed in problem 1.

7. The four-momentum operator P^μ is given by

$$P^\mu = \int d\mu(p)\, p^\mu\, a(p)^\dagger a(p) .$$

Show the following commutation relations:

$$[P^\mu, a(p)] = -p^\mu a(p), \quad \left[P^\mu, a(p)^\dagger\right] = p^\mu a(p)^\dagger .$$

8. Show:

$$\exp(iPa)\phi(x)\exp(-iPa) = \phi(x+a) .$$

Hint: It is sufficient to check the infinitesimal version of this relation.

9. Show:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(x-y)\phi(0)|0\rangle .$$

Hint: Use the formula of the previous problem.

10. The propagator of the (free) Klein-Gordon field is defined by

$$\Delta(x) = i\langle 0|T\phi(x)\phi(0)|0\rangle .$$

Show: $\Delta(-x) = \Delta(x)$.

11. Show that $\Delta(x)$ is a Green function of the Klein-Gordon operator, i.e.

$$(\Box + m^2)\Delta(x) = \delta^{(4)}(x) .$$

Discuss the behaviour of $\Delta(x)$ for positive (negative) x^0.