Hoofdstuk 2

Propagators, scattering theory and n-point functions.

2.1 Non-relativistic propagator theory.

The evolution operator in the Schrödinger picture $\hat{U}(t,t')$ satisfies

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)\hat{U}(t,t') = 0 \tag{2.1a}$$

$$\lim_{t'\to t}\hat{U}(t,t') = \mathbb{1} \ . \tag{2.1b}$$

If we wish to implement causality, we must impose the condition $t > t'$. We therefore introduce the operator

$$G(t,t') = -i\theta(t-t')\hat{U}(t,t') \tag{2.2}$$

which describes propagation forward in time and which we shall call the *causal propagator*. In the configuration representation we have

$$G(\vec{x},t;\vec{x}',t') = -i\theta(t-t')\langle\vec{x}|\hat{U}(t,t')|\vec{x}'\rangle \ , \tag{2.3}$$

and this propagator gives the probability amplitude that a particle which you detect at t' in $\vec{x}'$ shall be detected later at t in $\vec{x}$. Since

$$\frac{\partial}{\partial t}\theta(t-t') = \delta(t-t') \tag{2.4a}$$

$$\langle\vec{x}|\mathbb{1}|\vec{x}'\rangle = \delta^{(3)}(\vec{x}-\vec{x}') \tag{2.4b}$$

the causal propagator obeys

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)G(\vec{x},t;\vec{x}',t') = \hbar\delta^{(4)}(x-x') \tag{2.5a}$$

$$G(\vec{x},t;\vec{x}',t') = 0 \ , \qquad t < t' \ . \tag{2.5b}$$

From the definition of the evolution operator it follows that

$$\psi(\vec{x},t) = i\int d^3x'\; G(\vec{x},t;\vec{x}',t')\psi(\vec{x}',t')\;, \qquad t>t'\;, \tag{2.6}$$

which is the mathematical formulation of *Huygens' principle* for wave functions in quantum mechanics.

For a Hamiltonian of the form

$$\begin{aligned} \hat{H} &= \hat{H}_0 + V(\vec{x},t) \\ &= \frac{\hat{\vec{p}}^2}{2m} + V(\vec{x},t) \end{aligned} \tag{2.7}$$

we can write (2.5) as

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_0\right) G(\vec{x},t;\vec{x}',t') = \hbar\delta^{(4)}(x-x') + V(\vec{x},t)G(\vec{x},t;\vec{x}',t')\;. \tag{2.8}$$

Let $G_0(t,t') = -i\theta(t-t')\hat{U}_0(t,t')$ be the free causal propagator which obeys

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_0\right) G_0(\vec{x},t;\vec{x}',t') = \hbar\delta^{(4)}(x-x') \tag{2.9a}$$

$$G_0(\vec{x},t;\vec{x}',t') = 0\;, \qquad t<t'\;, \tag{2.9b}$$

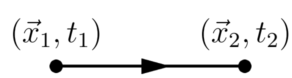
then we can write the solution of the propagator equation (2.5) as

$$G(\vec{x},t;\vec{x}',t') = G_0(\vec{x},t;\vec{x}',t') + \frac{1}{\hbar}\int d^4x_1\; G_0(\vec{x},t;\vec{x}_1,t_1)V(\vec{x}_1,t_1)G(\vec{x}_1,t_1;\vec{x}',t')\;. \tag{2.10}$$

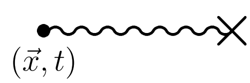
Through iteration we can write the previous equation as

$$\begin{aligned} G(\vec{x},t;\vec{x}',t') &= G_0(\vec{x},t;\vec{x}',t') + \frac{1}{\hbar}\int d^4x_1\; G_0(\vec{x},t;\vec{x}_1,t_1)V(\vec{x}_1,t_1)G_0(\vec{x}_1,t_1;\vec{x}',t') \\ &+ \frac{1}{\hbar^2}\iint d^4x_1 d^4x_2\; G_0(\vec{x},t;\vec{x}_1,t_1)V(\vec{x},t_1)G_0(\vec{x}_1,t_1;\vec{x}_2,t_2)V(\vec{x}_2,t_2)G_0(\vec{x}_2,t_2;\vec{x}',t') \\ &+ \cdots\;. \end{aligned} \tag{2.11}$$

We can represent this perturbation series diagrammatically by introducing the following correspondence rules:

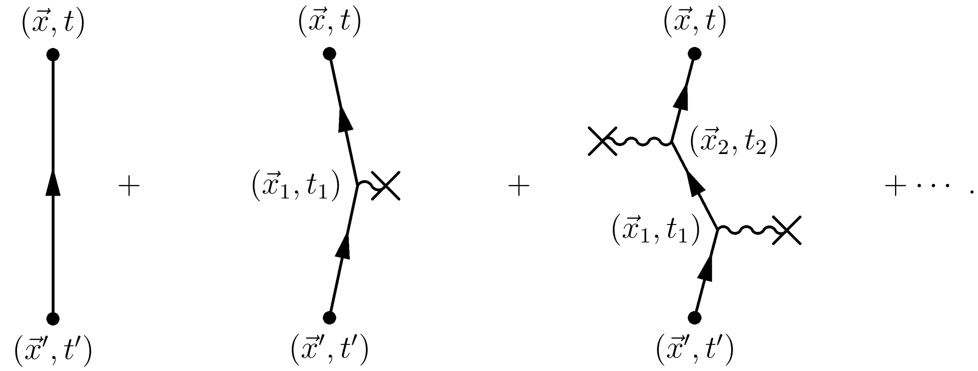
$G_0(\vec{x}_1,t_1;\vec{x}_2,t_2) =$ 

, (2.12a)

$\frac{1}{\hbar}V(\vec{x},t) =$ 

. (2.12b)

The perturbation series is then diagrammatically equal to



(2.13)

This diagrammatical representation of the perturbation series allows us to make a simple physical interpretation of every term. The first diagram represents free propagation from $(\vec{x}', t')$ to $(\vec{x}, t)$. We assume that time travels upwards so that there is causal ordering in this direction. The second diagram represents a particle that propagates from $(\vec{x}', t')$ to $(\vec{x}_1, t_1)$; there the particle is scattered by the potential$V(\vec{x}_1, t_1)$, and after that propagates further up to $(\vec{x}, t)$. Because of the superposition principle, we have to integrate over all $\vec{x}_1$ en t_1 where the scattering occurs. Notice that the free propagators which obey (2.9) automatically ensure that $t' < t_1 < t$ so that causality is respected. The third diagram represents a contribution to the propagation amplitude from $(\vec{x}', t')$ to $(\vec{x}, t)$ with intermediary scatterings at $(\vec{x}_1, t_1)$ and $(\vec{x}_2, t_2)$. It is obvious that the exact propagator can be obtained by summing over all possible ways of scattering between $(\vec{x}', t')$ and $(\vec{x}, t)$, what is precisely the diagrammatical transcription of the perturbation series (**??**). The advantage of the diagrammatical representation is that instead of a formal solution of the Schrödinger equation by means of a perturbation series, we now have a much more intuitive way of generating this same series by simply drawing all possible physical scattering processes and translating them into formulas using the correspondence rules(2.12).

2.2 Non-relativistic scattering theory: the Born series

Consider a non-relativistic particle which scatters from a heavy particle at rest. We assume this second particle to be heavy enough so that we can describe this process simply as potential scattering in a time independent potential$V(\vec{x})$, so that its wavefunction obeys

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = \left[-\frac{\hbar^2}{2m}\Delta + V \right] \psi \ . \tag{2.14}$$

We assume that $V(\vec{x})$ has short range so that the particle can be considered to be free, far from the scatterer, and to eveolve according to the free Hamiltonian. Because we are only interested in how the state of the particle changes due to the scattering process, we define the scattering operator as:

$$\begin{aligned}\hat{S} &= \lim_{\substack{t\to+\infty\\ t'\to-\infty}} e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{U}(t,t')e^{-\frac{i}{\hbar}\hat{H}_0 t'} \\ &= \lim_{\substack{t\to+\infty\\ t'\to-\infty}} e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}(t-t')}e^{-\frac{i}{\hbar}\hat{H}_0 t'} \\ &= \lim_{\substack{t\to+\infty\\ t'\to-\infty}} \hat{U}_I(t,t')\ , \end{aligned} \tag{2.15}$$

with $\hat{U}_I$ the evolution operator in the interaction picture. Using the definitions of the causal propagators $\hat{G}$ en $\hat{G}_0$, this can be rewritten as:

$$\hat{S} = -i \lim_{\substack{t\to+\infty\\ t'\to-\infty}} \hat{G}_0^{-1}(t,0)\hat{G}(t,t')\hat{G}_0^{-1}(0,t')\ . \tag{2.16}$$

This formula can be interpreted in the following way. The particle that is scattered by the potential V is coming in from asymptotically far in the past ($t'\to-\infty$) and also in space and reaches the short range potential (a heavy particle at rest) around $t\sim 0$ en $\vec{x}\sim\vec{0}$. Because we are not interested in the free propagation from $t'=-\infty$ to $t'=0$ we eleminate this free propagation by $\hat{G}_0^{-1}(0,t')$, $t'\to-\infty$. Analogously for $\hat{G}_0^{-1}(t,0)$ with $t\to+\infty$. This elimination of free propagators at the start and end of the process is sometimes called *amputation*.. Diagrammatically (see equation (2.13)) the external propagators are amputated.

To amputate the external propagators in the asymptotic regions, we rewrite the perturbation series as

$$G(x,x') = G_0(x,x') + \frac{1}{\hbar}\int d^4y d^4z\ G_0(x,y)T(y,z)G_0(z,x') \tag{2.17}$$

where we introduced space time coordinates $x=(t,\vec{x})$. Comparing with the perturbation series (**??**) for the propagator we find the Born series for T:

$$\begin{aligned} T(y,z) &= V(y)\delta^{(4)}(y-z) + \frac{1}{\hbar}V(y)G_0(y,z)V(z) \\ &\quad + \frac{1}{\hbar^2}\int d^4x_1\ V(y)G_0(y,x_1)V(x_1)G_0(x_1,z)V(z) \\ &\quad + \frac{1}{\hbar^3}\int d^4x_1 d^4x_2\ V(y)G_0(y,x_1)V(x_1)G_0(x_1,x_2)V(x_2)G_0(x_2,z)V(z) \\ &\quad + \cdots\ , \end{aligned} \tag{2.18}$$

which after resummation can be written according to (**??**) as

$$T(y,z) = V(y)\delta^{(4)}(y-z) + \frac{1}{\hbar}V(y)G(y,z)V(z)\ . \tag{2.19}$$

Defining the $\hat{T}$-operator by

$$T(x,x') = \langle\vec{x}|\hat{T}(t,t')|\vec{x}'\rangle\ , \tag{2.20}$$

we find that $\hat{T}$ fulfills

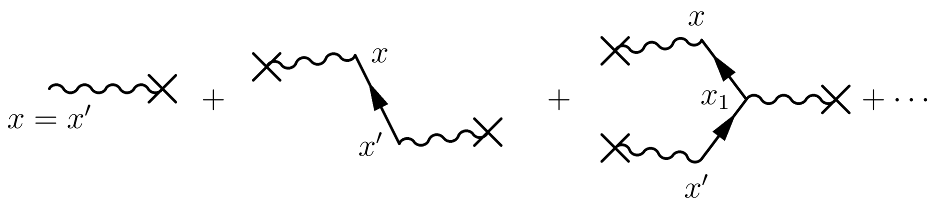
$$\hat{T}(t,t') = \hat{V}(t)\delta(t-t') + \frac{1}{\hbar}\hat{V}(t)\hat{G}(t,t')\hat{V}(t') \qquad (2.21)$$

and we can rewrite (2.17) as

$$\hat{G}(t,t') = \hat{G}_0(t,t') + \frac{1}{\hbar}\int dt_1 \int dt_2\ \hat{G}_0(t,t_1)\hat{T}(t_1,t_2)\hat{G}_0(t_2,t')\ . \qquad (2.22)$$

In the configuration representation, the $\hat{T}$-operator yields the amputated propagator $T(x,x') = \langle\vec{x}|\hat{T}(t,t')|\vec{x}'\rangle$, and has the following diagrammatical representation:

$T(x,x') =$



(2.23)

which can be obtained from equation (2.13) by amputating the asymptotic free propagators. The connection between the amputated propagator $\hat{T}$ and the S-matrix now follows from (**??**) and (2.22). Since for $t' < 0$ and $t > 0$ we have $\hat{G}_0(t_2,t') = \theta(t_2-t')e^{-\frac{i}{\hbar}\hat{H}_0 t_2}\hat{G}_0(0,t')$, $\hat{G}_0(t,t_1) = \theta(t-t_1)\hat{G}_0(t,0)e^{\frac{i}{\hbar}\hat{H}_0 t_1}$ and $\hat{G}_0(t,t') = i\hat{G}_0(t,0)\hat{G}(0,t')$ and we find

$$\begin{aligned}\hat{S} &= \hat{\mathbb{1}} - i \lim_{\substack{t\to+\infty \\ t'\to-\infty}} \frac{1}{\hbar}\int_{-\infty}^{t} dt_1 \int_{t'}^{+\infty} dt_2\ e^{\frac{i}{\hbar}\hat{H}_0 t_1}\hat{T}(t_1,t_2)e^{-\frac{i}{\hbar}\hat{H}_0 t_2} \\ &= \hat{\mathbb{1}} - \frac{i}{\hbar}\int dt_1 \int dt_2\ e^{\frac{i}{\hbar}\hat{H}_0 t_1}\hat{T}(t_1,t_2)e^{-\frac{i}{\hbar}\hat{H}_0 t_2}\ . \qquad (2.24)\end{aligned}$$

The transition amplitude $\langle f|\hat{S}|i\rangle$ that a particle in the initial state $|i\rangle$ scatters into the final state $|f\rangle$ is then:

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle - \frac{i}{\hbar}\int d_1 \int d_2\ e^{\frac{i}{\hbar}(E_f t_1 - E_i t_2)}\langle f|\hat{T}(t_1,t_2)|i\rangle\ . \qquad (2.25)$$

The non-trivial scattering is thus indeed given by the matrix elements of the $\hat{T}$-operator or amputated propagator. Introducing the wavefunctions $\psi_i(x) = \langle\vec{x}|i\rangle e^{-\frac{i}{\hbar}E_i t}$ and $\psi_f(x) = \langle\vec{x}|f\rangle e^{-\frac{i}{\hbar}E_f t}$ we can write the non-trivial scattering matrixelement as:

$$\langle f|\hat{S}|i\rangle_{\rm nt} = -\frac{i}{\hbar}\int d^4x d^4x'\ \psi_f^*(x)T(x,x')\psi_i(x')\ . \qquad (2.26)$$

This formula makes the connection between scattering and the amputated propagator very explicit and transparent.

For potentials which are independent of time, the integrals over t_1 en t_2 can be explicitely calculated. Because of translational invariance in the time direction, we have:

$$\hat{T}(t_1, t_2) = \hat{T}(t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} e^{-\frac{i}{\hbar}E(t_1 - t_2)} \hat{T}(E) \tag{2.27}$$

with $\hat{T}(E)$ the Fourier transform of $\hat{T}$. Substitution in (2.25) yields:

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle - 2\pi i \delta(E_f - E_i)\langle f|\hat{T}(E_i)|i\rangle \ . \tag{2.28}$$

Introducing also the Fourier transforms of $\hat{G}$ and $\hat{G}_0$:

$$\hat{G}(t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \hat{G}(E) e^{-\frac{i}{\hbar}E(t_1 - t_2)} \tag{2.29a}$$

$$\hat{G}_0(t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \hat{G}_0(E) e^{-\frac{i}{\hbar}E(t_1 - t_2)} \ , \tag{2.29b}$$

then (2.22) and (2.25) taking into account $\hat{V}(t) = \hat{V}$, become

$$\hat{G}(E) = \hat{G}_0(E) + \hat{G}_0(E)\hat{T}(E)\hat{G}_0(E) \tag{2.30}$$

and

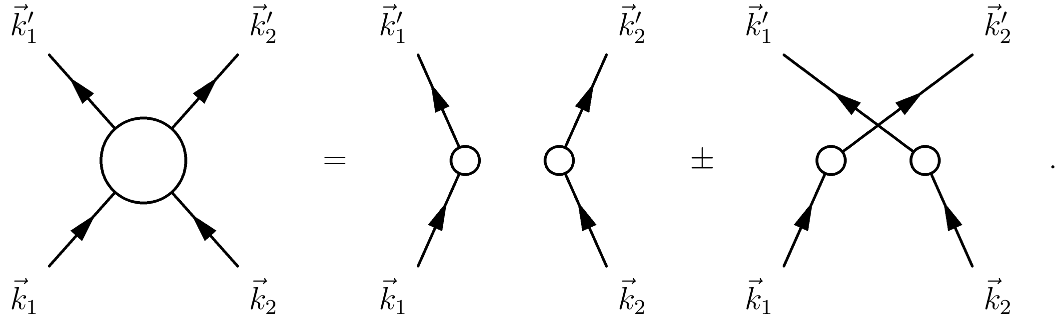
$$\hat{T}(E) = \hat{V} + \hat{V}\hat{G}(E)\hat{V} \ . \tag{2.31}$$

By iteration of these equations we finally obtain the Born series for the $\hat{T}$-matrix:

$$\hat{T}(E) = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \hat{V}\hat{G}_0\hat{V}\hat{G}_0\hat{V} + \cdots \ . \tag{2.32}$$

2.3 Non relativistic multi-particle scattering

When we consider scattering of several particles, a new aspect has to be taken into account: the indistinguishability of particles. Let's for example consider the scattering of two particles in a time independent potential$V(\vec{x})$. The incoming particles have momenta $\vec{k}_1$ and $\vec{k}_2$, the outgoing $\vec{k}_1'$ and $\vec{k}_2'$. Because of indistinguishability, there are two processes that can take place: the particle with $\vec{k}_1$ gets scattered to $\vec{k}_1'$, the one with $\vec{k}_2$ to $\vec{k}_2'$; and the *exchange*-proces: $\vec{k}_1$ gets scattered into $\vec{k}_2'$ and $\vec{k}_2$ into $\vec{k}_1'$. Diagrammatically:



(2.33)

The plus sign is for bosons, the minus sign for fermions — because of the symmetrical respectively antisymmetrical behaviour of the wavefunctions of bosons and fermions under permutation. From the diagram, it is also clear that the particles don't interact with each other and only with the potential $V(\vec{x})$ (whose T-matrix is represented by a circle). The T-matrix for two-particle scattering in an external potential is thus:

$$\langle\vec{k}_1',\vec{k}_2'|\hat{T}|\vec{k}_1,\vec{k}_2\rangle = \langle\vec{k}_1'|\hat{T}|\vec{k}_1\rangle\langle\vec{k}_2'|\hat{T}|\vec{k}_2\rangle \pm \langle\vec{k}_2'|\hat{T}|\vec{k}_1\rangle\langle\vec{k}_1'|\hat{T}|\vec{k}_2\rangle \; . \tag{2.34}$$

One can easily write down analogous formulas for n-particle scattering in an external potential. There is however an elegant way to automatically generate the correct exchange processes with appropriate signs: namely the introduction of field operators that create and annihilate particles. Indeed, we can rewrite the non-relativistic one-particle propagator as:

$$\begin{aligned} G(\vec{x},t;\vec{x}',t') &= -i\theta(t-t')\langle\vec{x},t|\vec{x}',t'\rangle \\ &= -i\theta(t-t')\langle\Theta|\psi(\vec{x},t)\psi^\dagger(\vec{x}',t')|\Theta\rangle \; . \end{aligned} \tag{2.35}$$

The causal propagation in time can now be absorbed in the definition of the time ordered product. Let $\hat{A}$ and $\hat{B}$ be two operators, then we define the time ordered product $T(\hat{A}(t)\hat{B}(t'))$ as:

$$\begin{aligned} T(\hat{A}(t)\hat{B}(t')) &= \begin{cases} \hat{A}(t)\hat{B}(t') \; , & t > t' \\ \pm\hat{B}(t')\hat{A}(t) \; , & t < t' \end{cases} \\ &= \theta(t-t')\hat{A}(t)\hat{B}(t') \pm \theta(t'-t)\hat{B}(t')\hat{A}(t) \; , \end{aligned} \tag{2.36}$$

where the plus sign is for bosonic operators and the minus sign for fermionic operators. Since in non-relativistic field theory there is conservation of the number of particles, we have

$$\psi(\vec{x},t)|\Theta\rangle = 0 \tag{2.37}$$

(ψ has only annihilation operators, see (1.205)) and we can write the causal propagator as:

$$G(\vec{x},t;\vec{x}',t') = -i\langle\Theta|T(\psi(\vec{x},t)\psi^\dagger(\vec{x}',t'))|\Theta\rangle \; . \tag{2.38}$$

The time ordered product has taken care of the restriction that only causal propagation is possible by, for $t < t'$, setting the field operator $\psi(\vec{x},t)$ to the right so that because of(2.37) the groundstate is annihilated and we get zero.

We can now ensure causal propagation in an elegant way for two particles by defining the following two particle propagator:

$$G(\vec{x}_1,t_1,\vec{x}_2,t_2;\vec{x}_1',t_1',\vec{x}_2',t_2') = (-i)^2\langle\Theta|T(\psi(\vec{x}_2,t_2)\psi(\vec{x}_1,t_1)\psi^\dagger(\vec{x}_1',t_1')\psi^\dagger(\vec{x}_2',t_2'))|\Theta\rangle \; . \tag{2.39}$$

Here the time ordered product for n bosonic operators is defined as:

$$T(\hat{A}_1(t_1)\hat{A}_2(t_2)\cdots\hat{A}_n(t_n)) = \hat{A}_{i_1}(t_{i_1})\hat{A}_{i_2}(t_{i_2})\cdots\hat{A}_{i_n}(t_{i_n}) \; , \qquad t_{i_1} > t_{i_2} > \cdots > t_{i_n} \tag{2.40}$$

and for n fermionic operators as:

$$T(\hat{A}_1(t_1)\hat{A}_2(t_2)\cdots\hat{A}_n(t_n)) = (-1)^P \hat{A}_{i_1}(t_{i_1})\hat{A}_{i_2}(t_{i_2})\cdots\hat{A}_{i_n}(t_{i_n}) \ , \qquad t_{i_1} > t_{i_2} > \cdots > t_{i_n} \tag{2.41}$$

with $i_1 i_2 i_3 \cdots i_n = P(123\cdots n)$ and P the permutation which brings the operators in chronological order with time augmenting from right to left. Notice that for fermions, every term is weighted by the parity of the permutation which brings the operators in the correct order. This parity factor automatically takes care of the desired symmetrical or antisymmetrical behaviour under permutation of particles:

$$T(\hat{A}_{P(1)}\hat{A}_{P(2)}\cdots\hat{A}_{P(n)}) = (\pm 1)^P T(\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n) \ . \tag{2.42}$$

Particularly, for the causal two-particle propagator:

$$\begin{aligned} G(\vec{x}_1, t_1, \vec{x}_2, t_2; \vec{x}_1', t_1', \vec{x}_2', t_2') &= \pm G(\vec{x}_2, t_2, \vec{x}_1, t_1; \vec{x}_1', t_1', \vec{x}_2', t_2') \\ &= \pm G(\vec{x}_1, t_1, \vec{x}_2, t_2; \vec{x}_2', t_2', \vec{x}_1', t_1') \ , \end{aligned} \tag{2.43}$$

where the minus sign is for fermions.

For further discussion, we will restrict ourselves to particles that move independently of each other in a time independent potential (free non-relativistic field theory). In this case, the calculation for causal multi-particle propagators reduces to the one of single particle propagators. Indeed in the case of two-particle scattering, the particles created at t_1' in $\vec{x}_1'$ and at t_2' in $\vec{x}_2'$ will propagate independently of each other to $\vec{x}_1$ at t_1 or to $\vec{x}_2$ at t_2. The time ordered product ensures causality: $t_1 > t_1'$, $t_2 > t_2'$ of $t_1 > t_2'$, $t_2 > t_1'$. Therefore there are two possibilities: the particle from $\vec{x}_1', t_1'$ propagates to $\vec{x}_1, t_1$ and the one from $\vec{x}_2', t_2'$ to $\vec{x}_2, t_2$, or the particle from $\vec{x}_1', t_1'$ propagates to $\vec{x}_2, t_2$ and the one from $\vec{x}_2', t_2'$ to $\vec{x}_1, t_1$. Thus we have:

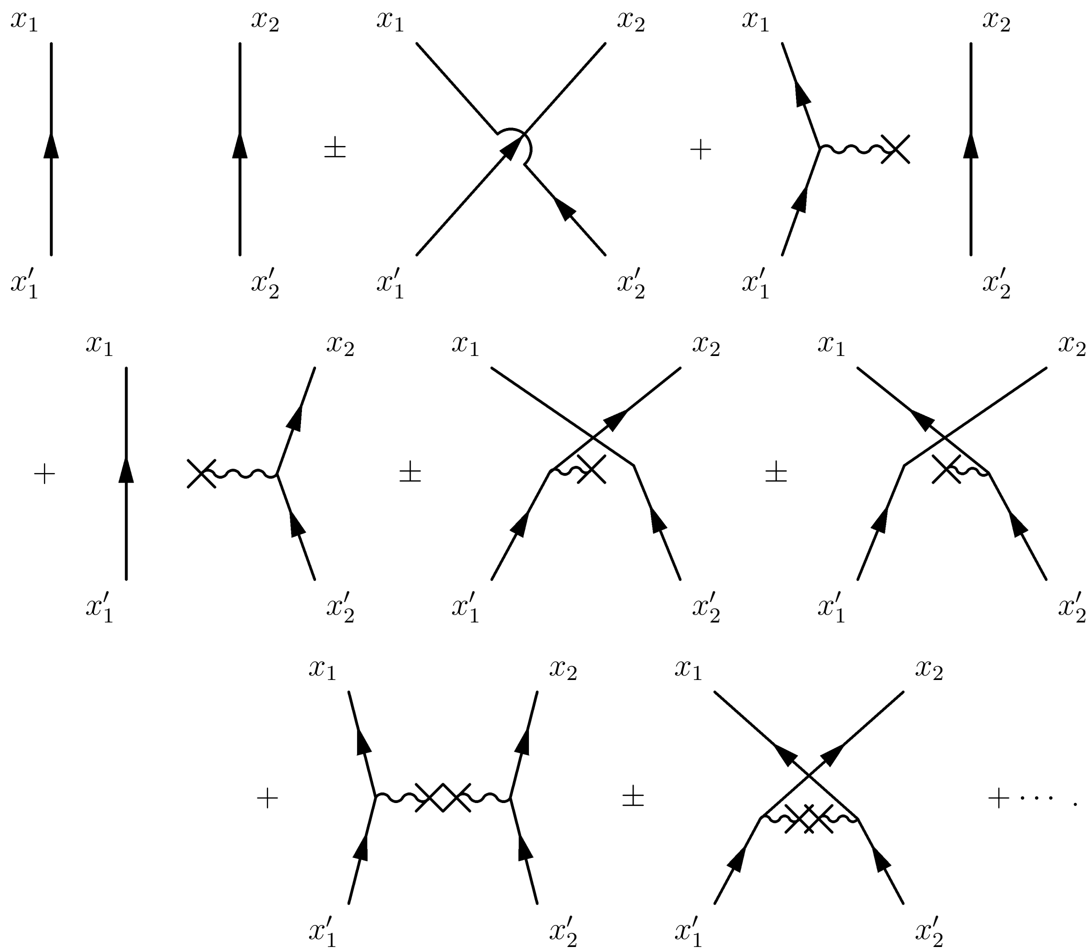
$$\begin{aligned} &\langle\Theta|T(\psi(\vec{x}_2, t_2)\psi(\vec{x}_1, t_1)\psi^\dagger(\vec{x}_1', t_1')\psi^\dagger(\vec{x}_2', t_2'))|\Theta\rangle \\ &\qquad = \langle\Theta|T(\psi(\vec{x}_1, t_1)\psi^\dagger(\vec{x}_1', t_1')|\Theta\rangle\langle\Theta|T(\psi(\vec{x}_2, t_2)\psi^\dagger(\vec{x}_2', t_2'))|\Theta\rangle \\ &\qquad\quad \pm \langle\Theta|T(\psi(\vec{x}_2, t_2)\psi^\dagger(\vec{x}_1', t_1')|\Theta\rangle\langle\Theta|T(\psi(\vec{x}_1, t_1)\psi^\dagger(\vec{x}_2', t_2'))|\Theta\rangle \ . \end{aligned} \tag{2.44}$$

The minus sign for fermions comes from the extra permutation that is necessary to put the operators in the correct time ordering.

From the definition (2.39) of the causal two-particle propagator, it follows then that:

$$\begin{aligned} G(\vec{x}_1, t_1, \vec{x}_2, t_2; \vec{x}_1', t_1', \vec{x}_2', t_2') &= G(\vec{x}_1, t_1; \vec{x}_1', t_1')G(\vec{x}_2, t_2; \vec{x}_2', t_2') \\ &\quad \pm G(\vec{x}_2, t_2; \vec{x}_1', t_1')G(\vec{x}_1, t_1; \vec{x}_2', t_2') \ . \end{aligned} \tag{2.45}$$

If we substitute the Born series for the one-particle propagators in the previous expression, we find diagrammatically for the two-particle propagator:



$$+\cdots \; . \tag{2.46}$$

Keeping only the non-trival terms where both particles scatter and amputating the external free propagators, we obtain the amputated two-particle propagator which in the case of two independently moving particles (free field) can be written in terms of the amputated one-particle propagators:

$$T(x_1, x_2; x_1', x_2') = T(x_1, x_1')T(x_2, x_2') \pm T(x_1, x_2')T(x_2, x_1') \; . \tag{2.47}$$

For the matrix elements we find:

$$\langle \vec{k}_1', \vec{k}_2' | \hat{T} | \vec{k}_1, \vec{k}_2 \rangle = \langle \vec{k}_1' | \hat{T} | \vec{k}_1 \rangle \langle \vec{k}_2' | \hat{T} | \vec{k}_2 \rangle \pm \langle \vec{k}_2' | \hat{T} | \vec{k}_1 \rangle \langle \vec{k}_1' | \hat{T} | \vec{k}_2 \rangle \; , \tag{2.48}$$

We can now define the causal n-particle propagator as:

$$\begin{aligned} &G(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \\ &\qquad = (-i)^n \langle \Theta | T(\psi(x_1)\psi(x_2)\cdots\psi(x_n)\psi^\dagger(y_1)\psi^\dagger(y_2)\cdots\psi^\dagger(y_n) | \Theta \rangle \; . \end{aligned} \tag{2.49}$$

This time ordered product represents the causal propagation of n particles from the space-time points $(y_1, y_2, \ldots, y_n)$ to $(x_1, x_2, \ldots, y_n)$. The expectation value in the ground state of a time ordered product of n field operators is called a n-point function. A n-particle propagator is hence a $2n$-point function. For free fields describing particles moving independently of each other we have Wick's theorem which says that:

$$G(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = \sum_P (\pm 1)^P G(x_{P(1)}, y_1) G(x_{P(2)}, y_2) \cdots G(x_{P(n)}, y_n) \quad (2.50)$$

with P a permutation of the particles and where the plus sign is for bosons, the minus sign for fermions. This is a straightforward generalization of the previous identity for two particles to n particles(no proof; for a proof see standard text books) and is a simple mathematical translation of the physical fact that the particles move independently and are indistinguishable (permutation symmetrie).

By keeping only the non-trivial terms in the Born series where all particles scatter and amputating the external free propagators, we obtain the amputated n-particle propagator. Non-trivial S-matrixelements can be obtained from this by :

$$\begin{aligned} &\langle f_1, f_2, \ldots, f_n | \hat{S} | i_1, i_2, \ldots, i_n \rangle_{\text{nt}} \\ &\quad = \left(-\frac{i}{\hbar}\right)^n \int d^4x_1 d^4x_2 \cdots d^4x_n d^4y_1 d^4y_2 \cdots d^4y_n \; \psi^*_{f_1}(x_1) \psi^*_{f_2}(x_2) \cdots \psi^*_{f_n}(x_n) \\ &\qquad T(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \psi_{i_1}(y_1) \psi_{i_2}(y_2) \cdots \psi_{i_n}(y_n) \; , \end{aligned} \quad (2.51)$$

which because of (**??**) can be rewritten as

$$\begin{aligned} &\langle f_1, f_2, \ldots, f_n | \hat{S} | i_1, i_2, \ldots, i_n \rangle_{\text{nt}} \\ &\quad = \frac{1}{\hbar^n} \int d^4x_1 d^4x_2 \cdots d^4x_n d^4y_1 d^4y_2 \cdots d^4y_n \; \psi^*_{f_1}(x_1) \psi^*_{f_2}(x_2) \cdots \psi^*_{f_n}(x_n) \\ &\qquad \langle \Theta | T(\psi(x_1)\psi(x_2) \cdots \psi(x_n) \psi^\dagger(y_1) \psi^\dagger(y_2) \cdots \psi^\dagger(y_n)) | \Theta \rangle_{\text{amp}} \\ &\qquad \psi_{i_1}(y_1) \psi_{i_2}(y_2) \cdots \psi_{i_n}(y_n) \; . \end{aligned} \quad (2.52)$$

We have derived this formula for particles which move independently from each other in an external potential. Because of the generality of the formula (it only uses general quantum mechanical principles and the existence of asymptotic free particles and does not refer to the precise Hamiltonian of the field theory), it is also valid for more general scattering processes where the particles interact amongst themselves.

2.4 Relativistic one-particle and multi-particle propagators for spin 1/2

The causal relativistic propagator $K_F(\vec{x}, t; \vec{x}', t')$ for a spin-1/2-fermion satisfies the equation

$$\left(i\hbar \frac{\partial}{\partial t} - \hat{H}\right) K_F(\vec{x}, t; \vec{x}', t') = \hbar \delta^{(4)}(x - x') \quad (2.53)$$

with $\hat{H}$ the Dirac-Hamiltonian in an external electromagnetic field:

$$\hat{H} = c\vec{\alpha}.\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + eV \; . \tag{2.54}$$

We have deliberately not yet imposed any causal boundary conditions in time for K_F because these are not a priori clear for the relativistic case. For example, let's naively take the definition

$$\begin{aligned} K_F(\vec{x},t;\vec{x}',t') &= -i\theta(t-t')\langle\vec{x}|U(t,t')|\vec{x}'\rangle \\ &= -i\theta(t-t')\langle\vec{x}|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|\vec{x}'\rangle \\ &= -i\theta(t-t')\left[\sum_n \psi_n^{(+)}(\vec{x})\psi_n^{(+)*}(\vec{x}')e^{-\frac{i}{\hbar}E_n(t-t')}\right. \\ &\quad \left. + \sum_n \psi_n^{(-)}(\vec{x})\psi_n^{(-)*}(\vec{x}')e^{+\frac{i}{\hbar}E_n(t-t')}\right] \; , \end{aligned} \tag{2.55}$$

then positive energy fermions as well as negative energy fermions propagate causaly —i .e. forward in time. This means that negative-energy fermions behave as real particles. This is wrong because according to the hole-theory of Dirac, it is the removal of a negative energy particle which acts causally as a real particle with opposite charge. Feynman solved this problem by imposing anti-causal conditions for the negative energy fermions, i.e. they should move backward in time. Indeed, if $t < t'$ (anti-causal), the time dependence of negative-energy solutions in (**??**) can be rewritten as

$$e^{\frac{i}{\hbar}E_n(t-t')} = e^{-\frac{i}{\hbar}E_n(t'-t)} \tag{2.56}$$

and hence is the same as for positive energy solutions which move from t to t' and thus causally. Imposing anti-causal boundary conditions in time to negative-energy solutions also ensures that these solutions behave as postive-energy soltutions with opposite charge because a charge that runs backward in time looks like an opposite charge that runs forward in time.

The correct causal relativistic propagator with anti-causal conditions for the negative-energy solutions is then:

$$\begin{aligned} K_F(\vec{x},t;\vec{x}',t') = &- i\theta(t-t')\sum_n \psi_n^{(+)}(\vec{x})\psi_n^{(+)*}(\vec{x}')e^{-\frac{i}{\hbar}E_n(t-t')} \\ &+ i\theta(t'-t)\sum_n \psi_n^{(-)}(\vec{x})\psi_n^{(-)*}(\vec{x}')e^{\frac{i}{\hbar}E_n(t-t')} \; . \end{aligned} \tag{2.57}$$

Notice that the negative-energy solutions have a relative minus sign with respect to the positive-energy solutions because otherwise the propagator equation (2.53) would not be fulfilled. Indeed, from (**??**) and

$$\frac{\partial}{\partial t}\theta(t-t') = -\frac{\partial}{\partial t}\theta(t'-t) = \delta(t-t') \tag{2.58}$$

we obtain :

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right) K_F(\vec{x},t;\vec{x}',t') = \hbar\delta(t-t')\left[\sum_n \psi_n^{(+)}(\vec{x})\psi_n^{(+)*}(\vec{x}') + \sum_n \psi_n^{(-)}(\vec{x})\psi_n^{(-)*}(\vec{x}')\right] = \hbar\delta^{(4)}(x-x') \ , \tag{2.59}$$

where we used the completeness relation of the energy-eigenfunctions. The relativistic evolution equation for $\psi(\vec{x},t)$ is then:

$$\psi(\vec{x},t) = i\int d^3x' \ K_F(\vec{x},t;\vec{x}',-\infty)\psi(\vec{x}',-\infty) - i\int d^3x' \ K_F(\vec{x},t;\vec{x}',+\infty)\psi(\vec{x}',+\infty) \tag{2.60}$$

To solve the Dirac equation, we must impose initial conditions at $t=-\infty$ for the positive-energy part and final conditions at $t=+\infty$ for the negative-energy part corresponding with causal and anti-causal propagation.

We can write the propagator equation (2.53) in a manifestly Lorentz invariant way by multiplying the right hand side and the left hand side by γ_0 and dividing by c. Then because of (2.54) one obtains that

$$\left[i\hbar\left(\frac{\partial}{\partial x^0}\gamma^0 + \nabla_i\gamma^i\right) - \frac{e}{c}\left(V\gamma^0 + A_i\gamma^i - mc\right)\right] K_F(x;x')\gamma_0 = \hbar\delta^{(4)}(x-x') \tag{2.61}$$

or

$$\left(i\hbar\partial\!\!\!/ - \frac{e}{c}A\!\!\!/ - mc\right) S_F(x;x') = \hbar\delta^{(4)}(x-x') \tag{2.62}$$

with

$$S_F(x;x') = K_F(x;x')\gamma^0 = -i\theta(t-t')\sum_n \psi_n^{(+)}(\vec{x})\overline{\psi}_n^{(+)}(\vec{x}')e^{-\frac{i}{\hbar}E_n(t-t')} + i\theta(t'-t)\sum_n \psi_n^{(+)}(\vec{x})\overline{\psi}_n^{(+)}(\vec{x}')e^{\frac{i}{\hbar}E_n(t-t')} \ . \tag{2.63}$$

Notice that we use from now on the definition $\delta^{(4)}(x-x') = \delta(x_0-x_0')\delta^{(3)}(\vec{x}-\vec{x}') = \frac{1}{c}\delta(t-t')\delta^{(3)}(\vec{x}-\vec{x}')$. The factor $1/c$ in the right hand side of(2.61) and (2.62) is now absorbed in the Dirac-deltafunction.

The propagator $S_F(x;x')$ is called the *Feynmanpropagator*. The advantage of working with the Feynmanpropagator $S_F = K_F\gamma_0$ instead of the relativistic causal propagator K_F , is that S_F satisfies a manifestly Lorentz invariant equation (2.62) and that manifestly relativistically invariant perturbation theory is now possible . We can now in an analogous way as for the non-relativistic propagator, expand the Feynmanpropagator in a Born series. Indeed, we can rewrite (2.62) as:

$$(i\partial\!\!\!/ - m)S_F(x;x') = \delta^{(4)}(x-x') + eA\!\!\!/(x)S_F(x;x') \ , \tag{2.64}$$

where we introduced natural units$\hbar = c = 1$ hebben ingevoerd. By introducing the free Feynmanpropagator $S_F^0(x;x')$ which is a solution of

$$(i\partial\!\!\!/ - m)S_F^0(x;x') = \delta^{(4)}(x-x') \ , \tag{2.65}$$

we can rewrite (2.64) as an integral equation for S_F:

$$S_F(x;x') = S_F^0(x;x') + \int d^4x_1 \ S_F^0(x;x_1)eA\!\!\!/(x_1)S_F(x_1;x') \ . \tag{2.66}$$

By iteration, we find a solution in the form of a perturbation series:

$$\begin{aligned} S_F(x;x') &= S_F^0(x;x') + \int d^4x_1 \ S_F^0(x;x_1)eA\!\!\!/(x_1)S_F^0(x_1;x') \\ &+ \int d^4x_1 \int d^4x_2 \ S_F^0(x;x_1)eA\!\!\!/(x_1)S_F^0(x_1;x_2)eA\!\!\!/(x_2)S_F^0(x_2;x') + \cdots \ . \end{aligned} \tag{2.67}$$

In contradistinction with the non-relativistic case, now propagation is possible in both directions: forward in time (solutions with met positive energy) and backward in time (solutions with negative energy which act as anti-particles with positive energy). If increasing time runs upwards in the drawing, then we have four possibilities of time ordering to first order in the coupling constant$e$:

(a) (b) (c) (d)

Diagram (a) represents an electron that scatters in an external electromagnetic field. Diagram (b) represents the scattering of a negative-energy electron that runs backwards in time. We interpret this as a positron that runs forward in time and scatters in the electromagnetic field. Diagram (c) represents a negative-energy electron that comes from $\vec{x}', t'$, runs backward in time to $\vec{x}_1, t_1$ and there gets scattered forward in time by the external field to $\vec{x}, t$. We interpret this as the creation of an electron-positron pair by the external electromagnetic field in $\vec{x}_1, t_1$, which after that propagates respectively to $\vec{x}, t$ and $\vec{x}', t'$. Diagram (d) represents electron-positron annihilation by the external field at $\vec{x}_1, t_1$ voor. Notice that the arrow points in the direction in which the electron moves. Although a negative-energy electron moves backward in time as seen by the observer in the lab frame (according to time in the lab frame), the proper time τ of this electron is always running forward and his proper energy (measured in a frame fixed to the electron) is positive. The arrow on the propagator points therefore in the direction of increasing proper time of the elctron.

We can now, in the same way as for the non-relativistic case, define the amputated causal propagator$K_F^{\rm amp}$ by

$$K_F(x;x') = K_F^0(x;x') + \int d^4y \int d^4z \ K_F^0(x;y)K_F^{\rm amp}(y;z)K_F^0(z;x') \tag{2.68}$$

and from this the non-trivial S-matrix element

$$\langle f|\hat{S}|i\rangle_{\text{nt}} = -i \int d^4x \int d^4x' \; \psi_f^*(x) K_F^{\text{amp}}(x;x')\psi_i(x') \; . \tag{2.69}$$

Analogously we can define the amputated Feynmanpropagator T by

$$S_F(x;x') = S_F^0(x;x') + \int d^4y \int d^4z \; S_F^0(x;y)T(y;z)S_F^0(z;x') \; . \tag{2.70}$$

From $S_F = K_F\gamma^0$ follows that $T = \gamma^0 K_F^{\text{amp}}$, so that

$$\langle f|\hat{S}|i\rangle = -i\epsilon_f \int d^4x \int d^4y \; \overline{\psi}_f(x)T(x;x')\psi_i(x') \; , \tag{2.71}$$

what constitutes a manifestly invariant form of the matrix element. Because the connection between the evolution operator and the propagator for fermions has an extra minus sign for propagation backwards in time, the phase factor ϵ_f is equal to 1 for scattering forward in time and -1 for scattering backward in time.

For relativistic electron scattering we have the complication that electrons can move forward as well as backward in time so that it is not apriori clear what are initial and final states. This however can be solved quite simply by looking at the scattering process from the eigen frame of the electron or from the standpoint of an observer whose time is the proper timeτ of the electron. Then it is clear that the initial state correponds with $\tau = -\infty$ and the final state with $\tau = +\infty$, or to put it differently, the initial state is where the electron arrow starts, the final state iswhere it ends. Hence:

Electron-scattering $(\vec{p}, s) \to (\vec{p}', s')$, $\epsilon_f = 1$:

$$\psi_i(x) = \psi^{(+)}_{(\vec{p},s)}(\vec{x},t) = \sqrt{\frac{m}{E(p)}}\frac{1}{(2\pi)^{3/2}} u^s(p) e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)} \tag{2.72a}$$

$$\psi_f(x) = \psi^{(+)}_{(\vec{p}',s')}(\vec{x},t) = \sqrt{\frac{m}{E(p')}}\frac{1}{(2\pi)^{3/2}} u^{s'}(p') e^{\frac{i}{\hbar}(\vec{p}'\cdot\vec{x}-E't)} \; . \tag{2.72b}$$

Positron-scattering $(\vec{p}, s) \to (\vec{p}', s')$, $\epsilon_f = -1$:

$$\psi_i(x) = \psi^{(-)}_{(\vec{p},s)}(\vec{x},t) = \sqrt{\frac{m}{E(p)}}\frac{1}{(2\pi)^{3/2}} v^s(p) e^{-\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)} \tag{2.72c}$$

$$\psi_f(x) = \psi^{(-)}_{(\vec{p}',s')}(\vec{x},t) = \sqrt{\frac{m}{E(p')}}\frac{1}{(2\pi)^{3/2}} v^{s'}(p') e^{-\frac{i}{\hbar}(\vec{p}'\cdot\vec{x}-E't)} \; . \tag{2.72d}$$

Pair-creation of electron $(\vec{p}, s)$ and positron $(\vec{p}', s')$, $\epsilon_f = 1$:

$$\psi_i(x) = \psi^{(-)}_{(\vec{p}',s')}(x) \; , \qquad \psi_f(x) = \psi^{(+)}_{(\vec{p},s)}(x) \; . \tag{2.72e}$$

Figuur 2.1: At $\vec{x}, t$ an electron-positron pair is created by the external field

Annihilation of electron $(\vec{p}, s)$ and positron $(\vec{p}', s')$, $\epsilon_f = -1$:

$$\psi_i(x) = \psi^{(+)}_{(\vec{p},s)}(x) \ , \qquad \psi_f(x) = \psi^{(-)}_{(\vec{p}',s')}(x) \ . \tag{2.72f}$$

Indeed, take for example pair-creation of electron and positron. An electron starts at $t = +\infty$ $(\tau = -\infty)$ with momentum $-\vec{p}'$, helicity $-s'$ and energy $-E(p')$, then travels backward in time to $\vec{x}, t$. There it gets scattered, forward in time, to $t = +\infty$ $(\tau = +\infty)$ with momentum $\vec{p}$, helicity s and energy $E(p)$ (see figure 2.1). Therefore, the initial wave function $(\tau = -\infty)$ is the one of an electron with negative energy:

$$\psi^{(-)}_{(\vec{p}',s')}(\vec{x}, t) = \sqrt{\frac{m}{E(p')}} \frac{1}{(2\pi)^{3/2}} v^{s'}(p') e^{-\frac{i}{\hbar}(\vec{p}' \cdot \vec{x} - E't)} \ . \tag{2.73}$$

This electron has momentum $-\vec{p}'$ and because of (1.252) helicity$-s'$. The corresponding positron that moves in the opposite direction in time has momentum $\vec{p}'$ and helicity s'. The scattering is forward in time so that $\epsilon_f = 1$.

We can now, just as in the non-relativistic case, write the Feynmanpropagator S_F as an expectation value in the ground state of a time ordered product of field operators. Indeed, we can expand the Dirac-field operator in a complete set of eigenfunctions :

$$\hat{\psi}(\vec{x}, t) = \sum_i \left[\hat{a}_i \psi^{(+)}(\vec{x}) e^{-\frac{i}{\hbar}E_i t} + \hat{b}^\dagger_i \psi^{(-)}_i(\vec{x}) e^{\frac{i}{\hbar}E_i t} \right] \tag{2.74}$$

with $\psi^{(+)}_i$ and $\psi^{(-)}_i$ the eigen functions with positive respectively negative energy. We have

$$\begin{aligned}
\langle \Theta | T(\hat{\psi}(\vec{x}, t) \hat{\overline{\psi}}(\vec{x}', t')) | \Theta \rangle &= \theta(t - t') \sum_{i,j} \langle \Theta | \hat{a}_i \hat{a}^\dagger_j | \Theta \rangle \psi^{(+)}_i(\vec{x}) \overline{\psi}^{(+)}_j(\vec{x}') e^{-\frac{i}{\hbar}(E_i t - E_j t')} \\
&\quad - \theta(t' - t) \sum_{i,j} \langle \Theta | \hat{b}_i \hat{b}^\dagger_j | \Theta \rangle \psi^{(-)}_i(\vec{x}) \overline{\psi}^{(-)}_j(\vec{x}') e^{\frac{i}{\hbar}(E_i t - E_j t')} \\
&= \theta(t - t') \sum_i \psi^{(+)}_i(\vec{x}) \overline{\psi}^{(+)}_i(\vec{x}') e^{-\frac{i}{\hbar}E_i(t - t')} \\
&\quad - \theta(t' - t) \sum_i \psi^{(-)}_i(\vec{x}) \overline{\psi}^{(-)}_i(\vec{x}') e^{\frac{i}{\hbar}E_i(t - t')} \\
&= i S_F(x; x') \ .
\end{aligned} \tag{2.75}$$

By analogy with the non-relativistic case we define the Feynman-multi-particle propagator as a $2n$-point function:

$$\begin{aligned}
&S_F(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \\
&\qquad = (-i)^n \langle \Theta | T(\psi(x_1)\psi(x_2) \cdots \psi(x_n) \overline{\psi}(x_1) \overline{\psi}(x_2) \cdots \overline{\psi}(x_n) | \Theta \rangle \ .
\end{aligned} \tag{2.76}$$

After amputation of the free Feynmanpropagators we can calculate the S-matrix elements via:

$$\langle f_1, f_2, \ldots, f_n|\hat{S}|i_1, i_2, \ldots, i_n\rangle_{\rm nt} = \prod_i \epsilon_{f_i} \int d^4x_1 \cdots d^4x_n d^4y_1 \cdots d^4y_n \, \overline{\psi}_{f_1}(x_1) \cdots \overline{\psi}_{f_n}(x_n)$$
$$\langle\Theta|T(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n))|\Theta\rangle_{\rm amp}\psi_{i_1}(y_1)\cdots\psi_{i_n}(y_n) \; . \quad (2.77)$$

2.5 Two point functions in Fourier space

Let's use the integral representation of the θ-function:

$$\theta(\tau) = \frac{i}{2\pi}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\varepsilon} \; , \quad (2.78)$$

then we can write the non-relativistic one-particle propagator as

$$\begin{aligned} G(\vec{x}, t; \vec{x}', t') &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega + i\varepsilon}\langle\vec{x}|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|\vec{x}'\rangle \\ &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega + i\varepsilon}\sum_i u_i(\vec{x})u_i^*(\vec{x}')e^{-\frac{iE_i}{\hbar}(t-t')} \\ &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_i \frac{u_i(\vec{x})u_i^*(\vec{x}')}{\omega - \omega_i + i\varepsilon} \end{aligned} \quad (2.79)$$

where we substituted $\omega \to \omega - \omega_i$ with $\hbar\omega_i = E_i$. The parameter ε is infinitesimal and positive. For a free particle ($V = 0$) we have $i \to \vec{k}$, $\sum_i \to \int \frac{d^3k}{(2\pi)^3}$, $u_i(\vec{x}) \to e^{i\vec{k}\cdot\vec{x}}$ and the one-particle propagator becomes:

$$G_0(\vec{x}, t; \vec{x}', t') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\int \frac{d^3k}{(2\pi)^3}\frac{e^{i(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega(t-t'))}}{\omega - \omega(\vec{k}) + i\varepsilon} \quad (2.80)$$

with $\omega(\vec{k}) = \hbar k^2/2m$.

Let ϕ be a Klein–Gordon field which we can assume to be generally complex, then we can define an n-particle Feynmanpropagator via the $2n$-point function:

$$\Delta_F(x_1, \ldots, x_n; y_1, \ldots, y_n) = (-i)^n\langle\Theta|T(\phi(x_1)\cdots\phi(x_n)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n))|\Theta\rangle \; . \quad (2.81)$$

The free one-particle propagator is then

$$\Delta_F^0(x, x') = (-i)\langle\Theta|T(\phi(x)\phi^\dagger(x'))|\Theta\rangle \quad (2.82)$$

with

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(k)}}[a(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega(k)t)} + b^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega(k)t)} \quad (2.83)$$

so that

$$\Delta_F^0(x;x') = (-i)\left[\theta(t-t')\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega(k)}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(k)(t-t')} \right. \\ \left. +\theta(t'-t)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega(k)}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(k)(t-t')}\right] \ . \quad (2.84)$$

Using the integral representation of the θ-function we obtain

$$\Delta_F^0(x;x') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega(t-t')}}{2\omega(k)}\left[\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\omega-\omega(k)+i\varepsilon}+\frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{-\omega-\omega(k)+i\varepsilon}\right] \quad (2.85)$$

where we did the substitutions $\omega \to \omega - \omega(k)$ and $\omega \to -(\omega+\omega(k))$. By changing $\vec{k} \to -\vec{k}$ in the second term, we finally find

$$\begin{aligned} \Delta_F^0(x;x') &= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\omega^2-(\omega(k)-i\varepsilon)^2} \\ &= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\omega^2-\omega(k)^2+i\varepsilon} \\ &= \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-x')}}{k^2-m^2+i\varepsilon} \ . \end{aligned} \quad (2.86)$$

In the last line we introduced four-vector notation: $k = (k_0, \vec{k})$, $x = (x_0, \vec{x})$, $k \cdot x = k_0 x_0 - \vec{k}\cdot\vec{x}$ and used $\omega^2(k) = |\vec{k}|^2 + m^2$.

We will now prove that the free one-particle Feynman propagator for Klein–Gordon fields obeys:

$$(\Box_x + m^2)\Delta_F^0(x;x') = -\delta^{(4)}(x-x') \ . \quad (2.87)$$

Indeed, for a free particle we have translation invariance in space and time. Therefore we propose the following Fourier transform:

$$\Delta_F^0(x-x') = \int\frac{d^4k}{(2\pi)^4}\tilde{\Delta}_F^0(k)e^{-ik\cdot(x-x')} \ . \quad (2.88)$$

Then

$$(\Box_x + m^2)\Delta_F^0(x-x') = \int\frac{d^4k}{(2\pi)^4}(-k^2+m^2)\tilde{\Delta}_F^0(k)e^{-ik\cdot(x-x')} \quad (2.89)$$

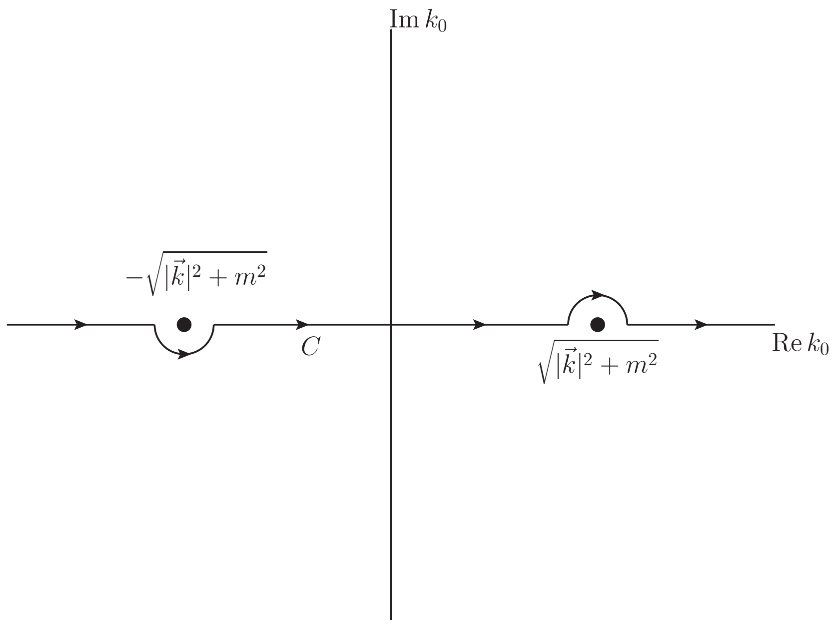
or because of $\delta^{(4)}(x-x') = \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}$:

$$\tilde{\Delta}_F^0(k) = \frac{1}{k^2-m^2} = \frac{1}{k_0^2-|\vec{k}|^2-m^2} \ . \quad (2.90)$$

The Fourier transform of Δ_F^0 has poles at

$$k_0 = \pm\sqrt{k^2+m^2} = \pm\omega(k) \ , \quad (2.91)$$

so that $\Delta_F^0(x-x')$ can not be unambiguously determined without specifying some contour in the complex plane around the poles. The Feynman propagator which describes particles with positive energy as moving forward in time and particles with negative energy as moving backward in time, can be obtained with the following choice of contour:



Indeed, let's only look at the k_0-integration. For $t > t'$ the Jordan lemma of contour integration is fulfilled if $\mathrm{Im}(k_0) < 0$, so we must close the contour in the lower halfplane where the pole $k_0 = +\omega(k)$ contributes because of the residue theorem:

$$\oint_C \frac{dk_2}{2\pi} \frac{e^{-ik_0(t-t')}}{k_0^2-\omega^2(k)} \underset{t>t'}{=} -2\pi i \frac{1}{2\pi} \frac{e^{-i\omega(k)(t-t')}}{2\omega(k)} \ . \tag{2.92}$$

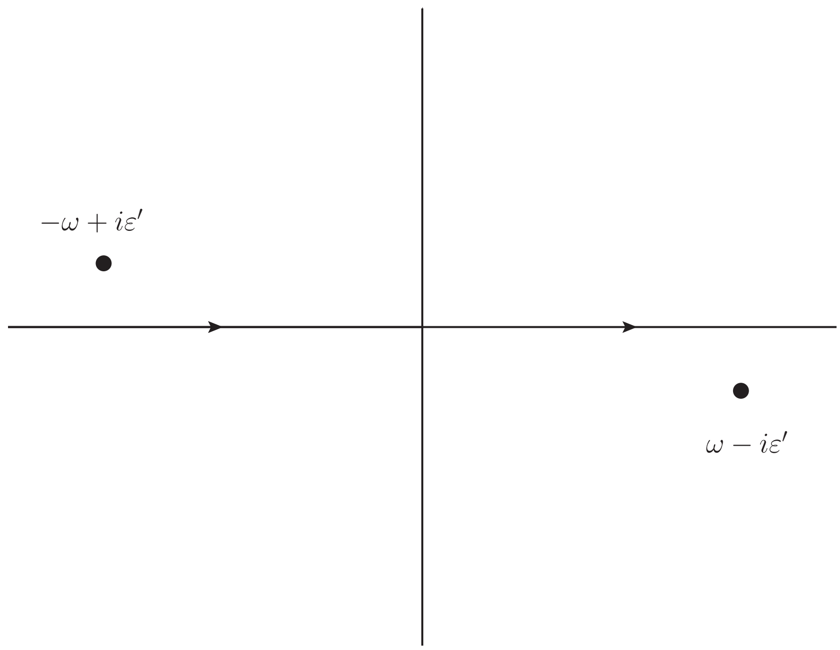
Only particles with positive energy propagate forward in time. For $t' > t$ we have to close the contour in the upper halfplane so that here only the pole $k_0 = -\omega(k)$ contributes:

$$\oint_C \frac{dk_2}{2\pi} \frac{e^{-ik_0(t-t')}}{k_0^2-\omega^2(k)} \underset{t'>t}{=} -2\pi i \frac{1}{2\pi} \frac{e^{+i\omega(k)(t-t')}}{2\omega(k)} \tag{2.93}$$

and only particles with negative energy propagate (backward in time). Adding both contributions we find

$$\Delta_F^0(x-x') = -i \left[\theta(t-t') \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(k)} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(k)(t-t')} \right.$$
$$\left. + \theta(t'-t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(k)} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{i\omega(k)(t-t')} \right] \tag{2.94}$$

what agrees with (**??**). We can now deform the contour by giving the positive energy pole an infinitesimally negative imaginary part and the pole with negative energy an infinitesimally positive imaginary part:



so that

$$k_0 = \pm(\omega(k) - i\varepsilon') \tag{2.95}$$

or

$$k_0^2 - |\vec{k}|^2 - m^2 + i\varepsilon = k^2 - m^2 + i\varepsilon = 0 \ . \tag{2.96}$$

The infinitesimal deformation of the contour and hence the implementation of the correct causal propagation can therefore be obtained by the simple substitution $m^2 \to m^2 - i\varepsilon$, and thus we find

$$\Delta_F^0(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{k^2 - m^2 + i\varepsilon} \tag{2.97}$$

which again agrees with (**??**).

In an analogous way, the Fourier representation of the Feynman-propagator of Dirac particles can be found. The free Feynmanpropagator S_F^0 satisfies

$$(i\partial\!\!\!/ - m) S_F^0(x - x') = \delta^{(4)}(x - x') \tag{2.98}$$

with

$$S_F^0(x - x') = \int \frac{d^4}{(2\pi)^4} e^{-ik\cdot(x-x')} \tilde{S}_F^0(k) \ . \tag{2.99}$$

By substitution we find that the Fouriertransform $\tilde{S}_F^0$ satisfies

$$(k\!\!\!/ - m)\tilde{S}_F^0(k) = 1 \tag{2.100}$$

or

$$\tilde{S}_F^0(k) = \frac{1}{k\!\!\!/ - m} = \frac{k\!\!\!/ + m}{k^2 - m^2} \ . \tag{2.101}$$

We can again implement causal propagation by the substitution $m^2 \to m^2 - i\varepsilon$ and finally obtain

$$S_F^0(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-x')} \; . \tag{2.102}$$

Analogously as for the Klein–Gordon propagator, we can show (exercise) that the Feynmanpropagator of the photon field in the Feynman gauge (Gupta-Bleuler with $\lambda = 1$) defined by

$$\Delta^0_{F\mu\nu}(x - x') = -i\langle\Theta|T(A_\mu(x)A_\nu(x'))|\Theta\rangle \tag{2.103}$$

satisfies

$$\Box\Delta^0_{F\mu\nu}(x - x') = g_{\mu\nu}\delta^{(4)}(x - x') \; . \tag{2.104}$$

After Fourier transformation we find

$$\Delta^0_{F\mu\nu}(x - x') = -g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{k^2 + i\varepsilon} \; . \tag{2.105}$$

For$\mu = i = 1, 2, 3$ this agrees exactly with the Klein–Gordon propagator of a massless particle.

2.6 Feynman rules for QED

Let's consider first the simple case of electron- (or positron-) scattering in an external electromagnetic field $A^{\text{ext}}_\mu(x)$. The Feynman rules which we will establish are the ones for $2n$-point functions (which differ with factors $-i$ from the n-particle propagators). We define diagrammatically:

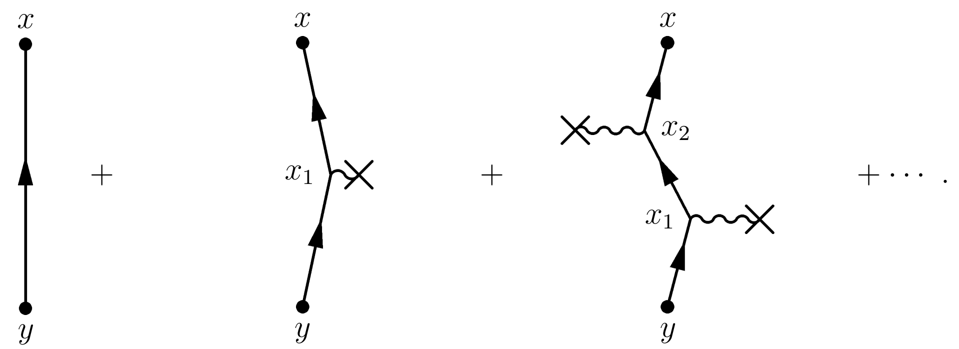
x  y $= \langle\Theta_0|T(\psi_0(x)\overline{\psi}_0(y))|\Theta_0\rangle = iS^0_F(x - y)$ (2.106a)

x  $= -ie\not{A}^{\text{ext}}(x)$ (2.106b)

with ψ_0 the free Dirac field and $|\Theta_0\rangle$ the corresponding vacuum state. Because of (2.66) , the two-point function $\langle\Theta|T(\psi(x)\overline{\psi}(y))|\Theta\rangle$ for a Dirac field in an external electromagnetic field A^{ext}_μ satiesfies the integral equation

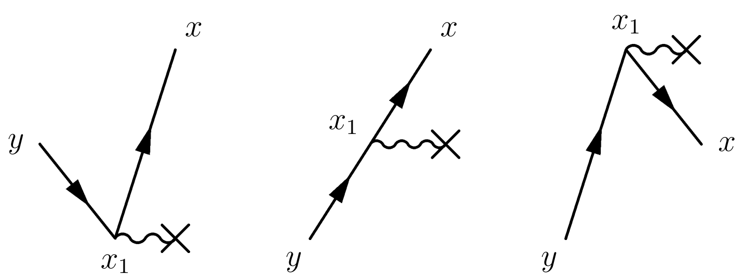
$$\begin{aligned}\langle\Theta|T(\psi(x)\overline{\psi}(y))|\Theta\rangle &= \langle\Theta_0|T(\psi_0(x)\overline{\psi}_0(y))|\Theta_0\rangle \\ &+ \int d^4x_1 \langle\Theta_0|T(\psi_0(x)\overline{\psi}_0(x_1))|\Theta_0\rangle(-ieA^{\text{ext}}_\mu(x_1))\langle\Theta|T(\psi(x_1)\overline{\psi}(y))|\Theta\rangle \; . \end{aligned} \tag{2.107}$$

By iteration one finds the diagrammatical perturbation series for the two-point function:



(2.108)

Notice that the arrows on the free propagators point in the direction of growing proper time of the electron. The end points x and y as well as the intermediary points x_1 and x_2 can be in each others future as well as past, as measured by a clock at rest in the lab frame. The integration over $x_1^0, x_2^0, \ldots$ is thus unrestricted. If $x_0 > y_0$ we can subdivide the integration over x_1^0 in $x_1^0 < y_0$, $y_0 < x_1^0 < x^0$ and $x_1^0 > x^0$. This corresponds with the following diagrams where the time in the lab frame flows upwards:



They represent respectively electron-positron pair creation , electron scattering and electron-positron pair annihilation. If $x_0 < y_0$ we have electron-positron-annihilation with $x \leftrightarrow y$, positron scattering and electron-positron-pair creation with $x \leftrightarrow y$. These six processes are described by one and the same Feynman diagram. The same analysis is valid for higher order Feynman diagrams.

Just as in the non-relativistic case we can consider electron-electron scattering as scattering of one electron in the external electromagnetic field generated by the other electron. In the Feynman gauge (Gupta-Bleuler with $\lambda = 1$) the photon fielsd obeys

$$\begin{aligned} \Box A_\mu &= j_\mu \\ &= e\overline{\psi}\gamma_\mu\psi \ . \end{aligned} \tag{2.109}$$

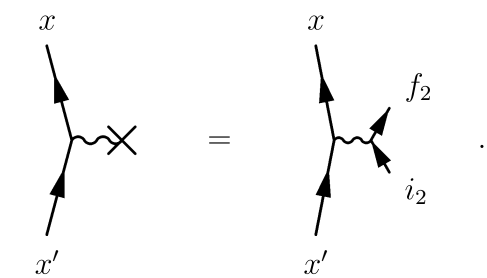
The unique solution which obeys causal propagation conditions is given by

$$A_\mu(x_1) = \int d^4x_2 \ \Delta^0_{F\mu\nu}(x_1 - x_2)e\overline{\psi}(x_2)\gamma^\nu\psi(x_2) \tag{2.110}$$

since the photon propagator $\Delta^0_{F\mu\nu}$ is because of (2.104) precisely the appropriate Green's function for this equation which implements the correct causal propagation. The electromagnetic field radiated by the second electron when it goes from the initial state $|i_2\rangle$ to the final state $|f_2\rangle$ is:

$$A_\mu(x_1) = \int d^4x_2 \; \Delta^0_{F\mu\nu}(x_1 - x_2) e\overline{\psi}_{f_2}(x_2)\gamma^\nu \psi_{i_2}(x_2) \; . \tag{2.111}$$

The corresponding Feynman diagram is in lowest order



(2.112)

If we take as initial state for the second electron, the state of a free electron that starts in the spacetime point y', and as final state, the state of a free electron that arrives in the spacetime point y, we have

$$\begin{aligned} \psi_{i_2}(x_2) &= \langle x_2|\overline{\psi}_0(y')|\Theta_0\rangle \\ &= \langle\Theta_0|T(\psi_0(x_2)\overline{\psi}_0(y'))|\Theta_0\rangle = iS^0_F(x_2 - y') \quad (2.113a) \\ \psi_{f_2}(x_2) &= \langle x_2|\overline{\psi}_0(y)|\Theta_0\rangle \quad (2.113b) \\ \Rightarrow \overline{\psi}_{f_2}(x_2) &= \langle\Theta_0|T(\psi_0(y)\overline{\psi}_0(x_2))|\Theta_0\rangle = iS^0_F(y - x_2) \; . \quad (2.113c) \end{aligned}$$

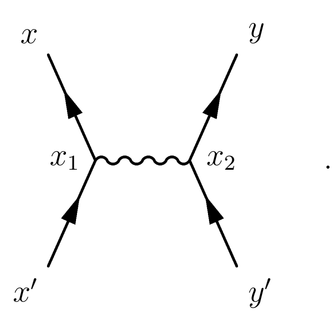
The external electromagnetic field is then because of (2.111) given by

$$A_\mu(x_1) = \int d^4x_2 (i\Delta^0_{F\mu\nu}(x_1 - x_2))(iS^0_F(y - x_2))(-ie\gamma^\nu)(iS^0_F(x_2 - y')) \; , \tag{2.114}$$

and by applying the Feynman rules for propagation in an external field we find for the two-point function:

$$\int d^4x_1 \int d^4x_2 \; (iS^0_F(x - x_1))(-ie\gamma^\mu)(iS^0_F(x_1 - x')) \\ i\Delta^0_{F\mu\nu}(x_1 - x_2)(iS^0_F(y - x_2))(-ie\gamma^\nu)(iS^0_f(x_2 - y')) \; . \tag{2.115}$$

We can represent this amplitude diagrammatically as

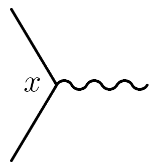


(2.116)

Herein is

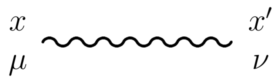
$$x = iS_F^0(x - x') \tag{2.117}$$

a diagrammatical representation of the electron propagator,



$$\mu = -ie\gamma^\mu \int d^4x \tag{2.118}$$

a diagrammatical representation of the elementary electron-photon interaction which we call the vertex and



$$= i\Delta^0_{F\mu\nu}(x - x') \tag{2.119}$$

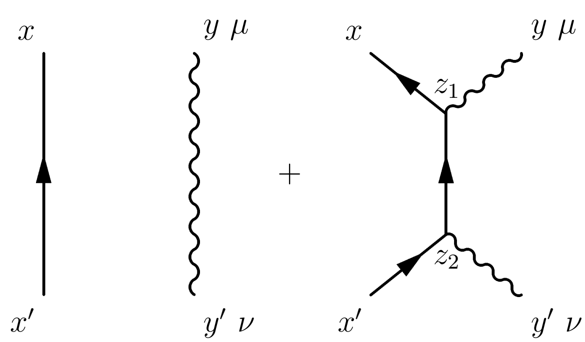
a diagrammatical representation of the photon propagator.

We can now give an elegant physical interpretation to this diagram. Two electrons start respectively in x' and y'. The first electron propagates freely to x_1, where it interacts by emitting a virtual photon. After the interaction, it propagates further to x. In the mean time the virtual photon propagates from x_1 to x_2, where it is absorbed by the second electron which propagates further to y. It is clear that we also have to add the *exchange*-diagram to get the total amplitude to order e^2 .

Up till now we restricted ourselves to electron scattering. But in the mean time the Feynman propagator for photons naturally appeared into our calculations, albeit as propagator for an intermediary photon which is exchanged between electrons in electron-electron scattering. But nothing stops us from using the same photon propagator to describe photon scattering. Compton scattering is the scattering of a photon by an electron. The two-particle propagator which describes this process is

$$\langle\Theta|T(\psi(x)A_\mu(y)\overline{\psi}(x')A_\nu(y'))|\Theta\rangle \ . \tag{2.120}$$

If we apply the Feynman rules, we have to order e^2:

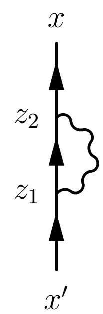


$$+ \cdots \ . \tag{2.121}$$

We may indeed use the same Feynman rules as for electron-electron scattering, because the electron that starts from x' and at z_2 absorbs a photon, does not make a difference between a virtual photon (that will be absorbed furtheron by an electron or positron or any other charged particle and in this way transmits momentum and hence causes scattering) and a real photon that was emitted from y'. The amplitude to order e^2 is thus

$$\int d^4z_1 \int d^4z_2 \; (i\Delta^0_{F\mu\alpha}(y-z_1))(iS^0_F(x-z_1))(-ie\gamma^\alpha)$$
$$(iS^0_F(z_1-z_2))(-ie\gamma^\beta)(iS^0_F(z_2-x'))(i\Delta^0_{F\beta\nu}(z_2-y')) \; . \quad (2.122)$$

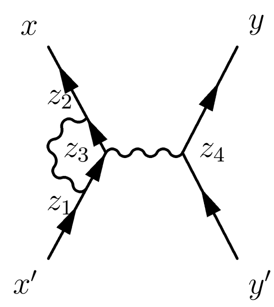
Let's now consider the following order e^2-diagram:



, (2.123)

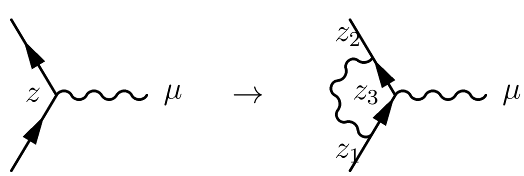
obtained by letting the emitted photon in the Compton scattering diagram be absorbed by the very same electron. This is an example of a self-energy diagram . Depending on the time ordering ($z^0_1 > z^0_2$ of $z^0_1 < z^0_2$) the electron absorbs at z_1 the photon that was emitted by the same electron at z_2 or vice versa. Hence the electron interacts with its own electromagnetic field and this diagram describes selfinteraction. It is clear that the electron does not get scattered in this diagram (its energy momentum vector is conserved). This self-energy diagram produces among other things an electromagnetic contribution to the mass of the electron. This is so called mass renormalisation.

Also the charge of the electron gets renormalised by quantum corrections. Consider the following order e^4 diagram for electron-electron scatteringdiagram :



. (2.124)

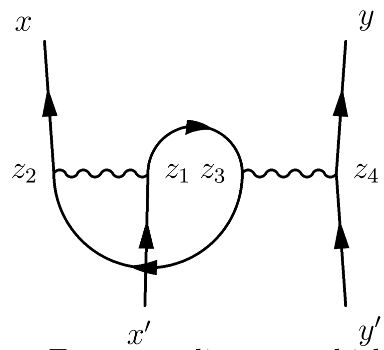
The extra photon propagator produces a vertex correction:



. (2.125)

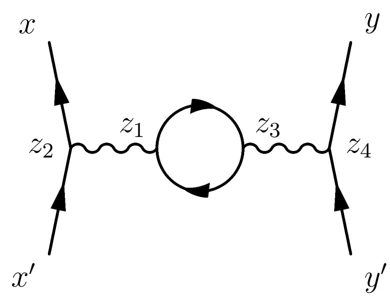
Because the vertex factor is $-ie\gamma_\mu \int d^4z_3$, the vertex correction generates a renormalisation of the elementary electron-photon interaction and hence also of the electron charge.

By continuous deformation, we can redraw the previous diagram as



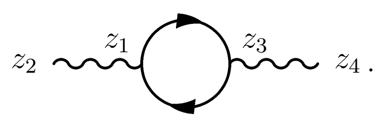
. (2.126)

This is topologically the same Feynman diagram which hence still represents the same process. Let us now cut loose , the electron propagators that arrive at z_2 (from z_3) and at z_1 (from x') at their endpoints, switch them (*exchange*), and stitch them back on. We then obtain the following diagram:



. (2.127)

We now see that the electrons exchange a virtual photon between z_2 and z_4 which contains a self-energy correction:

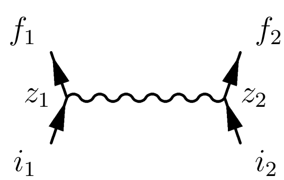


. (2.128)

This is a second example of a self-energy diagram and gauge invariance ensures that this self-energy does not renormalise the mass of the photon (it stays massles). Because the diagram was obtained from the previous one by exchange, there is an extra minus sign involved. This is an example of the general Feynman rule that every electron loop costs a minus sign.

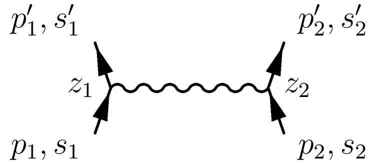
2.7 Feynman rules in momentum space

The Feynman rules in momentum space can be obtained by calculating the S-matrixelements for electron-electron and Compton scattering in lowest order. For electron-electron scattering we must according to the master formula(**??**) first amputate the external electron propagators and replace them with the initial and final wavefunctions of the electrons:



$-$ exchange . (2.129)

If we choose free electron states with good momentum and helicity, then this becomes:



$$- \text{ exchange } . \tag{2.130}$$

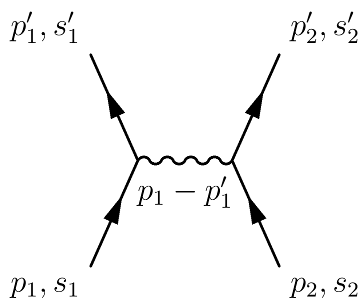
Applying the Feynman rules in configuration space to the amputated two-electron propagator and making use of (2.72), we find for the S-matrixelement in order e^2:

$$\begin{aligned}
&\int d^4z_1 \int d^4z_2\ \overline{\psi}_{p_1',s_1'}(z_1)(-ie\gamma^\mu)\psi_{p_1,s_1}(z_1)(i\Delta^0_{F\mu\nu}(z_1-z_2))\overline{\psi}_{p_2',s_2'}(z_2)(-ie\gamma^\nu)\psi_{p_2,s_2}(z_2) \\
&\qquad\qquad - (p_1', s_1' \leftrightarrow p_2', s_2') \\
&= \sqrt{\frac{m}{(2\pi)^3E(p_1')}}\sqrt{\frac{m}{(2\pi)^3E(p_1)}}\sqrt{\frac{m}{(2\pi)^3E(p_2')}}\sqrt{\frac{m}{(2\pi)^3E(p_2)}} \\
&\quad \times \bar{u}^{s_1'}(p_1')(-ie\gamma^\mu)u^{s_1}(p_1)\bar{u}^{s_2'}(p_2')(-ie\gamma^\nu)u^{s_2}(p_2) \\
&\quad \times \int d^4z_1 \int d^4z_1\ e^{i(p_1'-p_1)\cdot z_1}e^{i(p_2'-p_2)\cdot z_2} i\Delta^0_{F\mu\nu}(z_1-z_2)\ .
\end{aligned} \tag{2.131}$$

Using the Fourier representation of the photon propagator (2.105), then we find for the integration over z_1 and z_2:

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\int d^4z_1 \int d^4z_2\ e^{i(p_1'-p_1-k)\cdot z_1}e^{i(p_2'-p_2+k)\cdot z_2}\frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \\
&= (2\pi)^4\int d^4k\ \delta^{(4)}(p_1'-p_1-k)\delta^{(4)}(p_2'-p_2+l)\frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \\
&= (2\pi)^4\delta^{(4)}(p_2'+p_1'-(p_2+p_1))\frac{-ig_{\mu\nu}}{(p_1'-p_1)^2+i\varepsilon}\ .
\end{aligned} \tag{2.132}$$

The integration over z_1 took care of energy-momentum conservation in the first vertex en analogously the one over z_2 took care of energy-momentum conservation in the second vertex. After integration over the virtual photon momentum, it gets replaced by $p_1' - p_1$. The whole expression is proportional to $\delta^{(4)}((p_2'+p_1')-(p_2+p_1))$ which ensures conservation of total energy-momentum in the reaction. In the diagram, we can now drop the coordinates of the vertexpoints z_1 and z_2 because we integrated over them and replace the photon momentum by $p_1' - p_1$. We then obtain the Feynman diagram in momentum space:

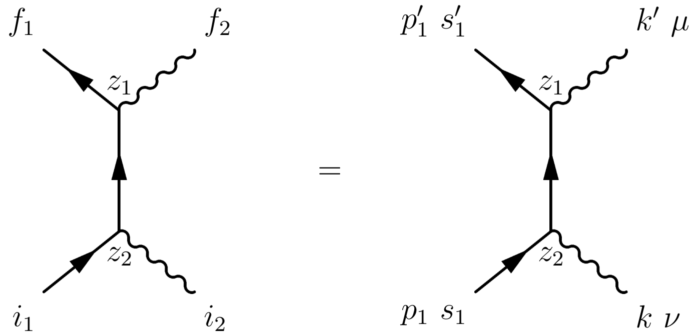


$$- (p_1', s_1' \leftrightarrow p_2', s_2')\ . \tag{2.133}$$

The corresponding Feynman amplitude is

$$
\begin{aligned}
(2\pi)^4\delta^{(4)}((p_2'+p_1')-(p_2+p_1))\sqrt{\frac{m}{(2\pi)^3E(p_1')}}\overline{u}^{s_1'}(p_1')(-ie\gamma^\mu)\sqrt{\frac{m}{(2\pi)^3E(p_1)}}u^{s_1}(p_1)& \\
\times\left(\frac{-ig_{\mu\nu}}{(p_1'-p_1)^2+i\varepsilon}\right)\sqrt{\frac{m}{(2\pi)^3E(p_2')}}\overline{u}^{s_2'}(p_2')(-ie\gamma^\nu)\sqrt{\frac{m}{(2\pi)^3E(p_2)}}u^{s_2}(p_2)& \\
-\,(p_1',s_1'\leftrightarrow p_2',s_2')\ .& \quad (2.134)
\end{aligned}
$$

We can do now the same thing for Compton scattering. We amputate the external propagators for electrons as well as for photons and replace them with initial and final wavefunctions:



. (2.135)

The initial and final wavefunctions for a photon with momentum k en polarisation λ, respectively k' and λ', are:

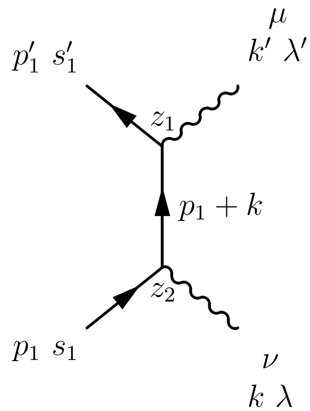
$$A_i^\mu(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega(k)}}e^{-ik\cdot x}\epsilon_\lambda^\mu(k) \quad (2.136a)$$

$$A_f^\mu(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega(k')}}e^{-ik'\cdot x}\epsilon_{\lambda'}^{\mu\,*}(k')\ . \quad (2.136b)$$

Because a photon is its own anti-particle (the photon field A^μ is real), we have only solutions with positive energy $k_0 = \omega(k)$ as incoming and outgoing wavefunctions. Furthermore, the polarisation of real photons is transversal so that $\lambda = 1, 2$. By amputation of the amplitude in configuration space (2.135) and substitution of the wavefunctions, we find for the S-matrixelement to order e^2:

$$
\begin{aligned}
\frac{1}{\sqrt{(2\pi)^3 2\omega(k')}}\sqrt{\frac{m}{(2\pi)^3E(p_1')}}\frac{1}{\sqrt{(2\pi)^3 2\omega(k)}}\sqrt{\frac{m}{(2\pi)^3E(p_1)}}& \\
\times\int d^4z_1\int d^4z_1\ e^{i(k'+p_1')\cdot z_1}\epsilon_{\lambda'}^{\mu\,*}(k')\overline{u}^{s_1'}(p_1')(-ie\gamma_\mu)& \\
iS_F^0(z_1-z_2)(-ie\gamma_\nu)u^{s_1}(p_1)\epsilon_\lambda^\nu(k)e^{-i(k+p_1)\cdot z_2}\ .& \quad (2.137)
\end{aligned}
$$

To carry out the integrations over z_1 and z_2 , we use the Fourier representation (2.102) for the electron propagator $S_F^0(z_1 - z_2)$. The integrations over z_1 and z_2 again ensure conservation of energy-momentum in the vertices, so that the Feynman diagram now becomes:



(2.138)

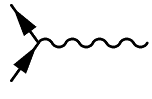
with corresponding amplitude:

$$(2\pi)^4\delta^{(4)}((p'_1+k')-(p_1+k))\frac{1}{\sqrt{(2\pi)^3 2\omega(k')}}\epsilon^{\mu\,*}_{\lambda'}(k')\sqrt{\frac{m}{(2\pi)^3E(p'_1)}}$$
$$\times\,\bar{u}^{s'_1}(p'_1)(-ie\gamma_\mu)\left(i\frac{(\not{p}_1+\not{k})+m}{(p_1+k)^2+m^2-i\epsilon}\right)(-ie\gamma_\nu)u^{s_1}(p_1)$$
$$\times\sqrt{\frac{m}{(2\pi)^3E(p_1)}}\epsilon^{\nu}_{\lambda}(k)\frac{1}{\sqrt{(2\pi)^3 2\omega(k)}}\ . \quad (2.139)$$

From these two examples we can now distill the Feynman rules in momentum space for a general nth-order diagram:

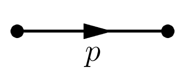
1. Draw all topologically inequivalent diagrams with n vertices that connect the external lines (initial- and final states) through the vertices and the internal lines (free photon- and electron propagators) together to a connected diagram.

2. Do for every vertex the substitution



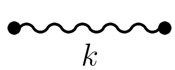
$$\mu\ \ \rightarrow -ie\gamma^\mu\ . \quad (2.140a)$$

Do for every internal electron line the substitution



$$\rightarrow \frac{i(\not{p}+m)}{p^2+m^2-i\varepsilon} \quad (2.140b)$$

and for every internal photon line the substitution



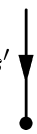
$$\mu \quad \nu \rightarrow \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}\ . \quad (2.140c)$$

3. Do for the external lines the substitutions :

–incoming electron:  $p, s \to \sqrt{\frac{m}{(2\pi)^3 E(p)}}\, u^s(p)$ (2.140d)

–final electron:  $p', s' \to \sqrt{\frac{m}{(2\pi)^3 E(p')}}\, \bar{u}^{s'}(p')$ (2.140e)

–incoming positron: $-p, s$  $\to \sqrt{\frac{m}{(2\pi)^3 E(p)}}\, \bar{v}^s(p)$ (2.140f)

–final positron: $-p', s'$  $\to \sqrt{\frac{m}{(2\pi)^3 E(p')}}\, v^{s'}(p')$ (2.140g)

–incoming photon: 

$k, \lambda \to \frac{1}{(2\pi)^3 2\omega(k)} \epsilon^\mu_\lambda(k)$ (2.140h)

–final photon:  $k', \lambda' \to \frac{1}{(2\pi)^3 2\omega(k')} \epsilon^{\mu\,*}_{\lambda'}(k')$ (2.140i)

4. If there are V vertices and I internal lines, there are V Dirac-delta functions that express conservation of energy-momentum in every vertex. By integrating over the momenta of the internal lines, $V-1$ internal momenta can be eliminated in terms of the external momenta and $L = I-V+1$ independent internal momenta will remain.These are the so called loop momenta because using the Euler theorem from graph theory one can show that there are as many independent internal momenta to be integrated over as ther are independent loops in the diagram. Every loop integration comes with a factor $1/(2\pi)^4$. The final remaining Dirac-delta function ensures overall conservation of energy and momentum and gives a factor $(2\pi)^4\delta^{(4)}(\sum_f p_f - \sum_i p_i)$.

5. Finally we must still calculate the overall sign. This sign is the product of

   (a) a factor (-1) for every incoming positron (an incoming positron is a final negative energy electron that is scattered backward in time so that $\epsilon_f = -1$ in (**??**)),

   (b) a factor (-1) for every closed electron loop,

   (c) a factor (-1) for every exchange of external electrons or positrons.