17. Spontaneous symmetry breaking

symmetries in QFT:

symmetry group G of QFT, i.e. $\{U(g) | g \in G\}$ is a representation of G in the Hilbert space of state vectors with $[U(g), H] = 0 \ \forall g \in G$

Wigner-Weyl realization of a symmetry:

multiplets with respect to G

$$H|\alpha\rangle = E|\alpha\rangle$$

eigenspace associated with eigenvalue E spanned by vectors $\{|\alpha\rangle\}$

$$HU(g)|\alpha\rangle = U(g)\underbrace{H|\alpha\rangle}_{E|\alpha\rangle} = E\,U(g)|\alpha\rangle$$

$$U(g)|\alpha\rangle = \sum_\beta |\beta\rangle \underbrace{\langle\beta|U(g)|\alpha\rangle}_{U_{\beta\alpha}(g)}$$

$\Rightarrow$ vectors $\{|\alpha\rangle\}$ form an invariant subspace under G (multiplet)

remark: this situation is well known from QM

potential invariant under rotations $\rightarrow$ angular momentum conserved $\rightarrow$ degenerate energy levels for given total angular momentum

if ground state unique $\Rightarrow U(g)|0\rangle = |0\rangle$ $\forall\, g \in G$ (ground state invariant under G)

basic assumption in QFT: unique vacuum state $|0\rangle$ ($P^\mu |0\rangle = 0$)

Nambu–Goldstone realization of a symmetry:

ground state not invariant under the symmetry group of the Hamiltonian examples from solid state physics:

ferromagnet: Hamiltonian invariant under rotations
but: for $T < T_c$ (Curie temperature) ground state not rotation invariant; no orientation of the spins energetically preferred, but a specific one singled out in the ground state → rotational symmetry spontaneously broken

infinite crystalline solid: translational symmetry spontaneously broken

assumption: theory (Hamiltonian) invariant under Lie group G (continuous group)

$$\partial_\mu J^\mu(x) = 0 \quad \rightarrow \quad Q = \int d^3x \; J^0(x)$$

$$\| Q|0\rangle \|^2 = \langle 0| Q\, Q |0\rangle =$$

$$= \int d^3x \; \langle 0| Q\, J^0(x) |0\rangle$$

$$= \int d^3x \; \langle 0| Q\, e^{iPx} J^0(0)\, e^{-iPx} |0\rangle$$

$$= \int d^3x \; \underbrace{\langle 0| Q\, J^0(0) |0\rangle}_{\text{independent of } x}$$

$$\Downarrow$$

Goldstone alternative

$Q|0\rangle = 0$	$\| Q|0\rangle \| = \infty$
	(i.e. Q not defined)
Wigner – Weyl	Nambu – Goldstone
linear representation of the symmetry group	nonlinear realization of the symmetry
degenerate multiplets	massless Goldstone bosons
unbroken (exact) symmetry	SSB

Goldstone theorem

basic assumption: theory invariant under Lie group G with conserved currents $J^\mu_a(x)$, $\partial_\mu J^\mu_a(x)=0$

as seen previously, the associated charges

$Q_a = \int d^3x \, J^0_a(x)$ are formal objects

causality ensures, however, that

$$[J^0_a(x), \underbrace{O(y)}_{\text{local operator}}] = 0 \quad \text{if } \underbrace{(x-y)^2<0}_{\text{space-like}}$$

$$\Rightarrow \quad \int d^3x \, [J^0_a(x), O(y)] = [Q_a, O(y)]$$

receives only contributions from the finite volume $|\vec{x}-\vec{y}| \leq |x^0-y^0|$

commutation relations $[Q_a, Q_b] = i f_{abc} Q_c$ are also of formal nature, but both sides of

$$[Q_a, J^\mu_b(x)] = i f_{abc} J^\mu_c(x)$$

may be applied to physical states, with a perfectly meaningful result

consider all <u>local</u> operators $O_m(x)$ that may be formed with the dynamical variables of the theory, their derivatives and their products at a given point x of space-time, allowing for an arbitrary, but finite number of derivatives and fields

transform according to representation $D(g)$ under G:

$$O_m \xrightarrow{g\in G} D_{mn}(g)\, O_n$$

$$e^{i\alpha_a Q_a} O_m e^{-i\alpha_b Q_b} = O_m + i\alpha_a [Q_a, O_m] + \mathcal{O}(\alpha^2)$$

$$= \left(e^{-i\alpha_a T_a}\right)_{mn} O_n$$

$$= O_m - i\alpha_a (T_a)_{mn} O_n + \mathcal{O}(\alpha^2)$$

$$[Q_a, O_m] = -(T_a)_{mn} O_n$$

$$[T_a, T_b] = i f_{abc} T_c$$

1) <u>Wigner-Weyl</u> : $|0\rangle$ invariant under G,

i.e. $Q_a|0\rangle = 0$

$$\underbrace{\langle 0| e^{i\alpha.Q} O_m e^{-i\alpha.Q} |0\rangle}_{\overset{Q_a|0\rangle=0}{\Rightarrow}\ \langle 0|O_m|0\rangle} = \left(e^{-i\alpha.T}\right)_{mn} \langle 0|O_n|0\rangle$$

only <u>those</u> operators, which are <u>invariant under</u> G, may acquire a <u>vaccum expectation value</u>

2) Nambu–Goldstone (SSB):

$\langle 0 | O_m | 0 \rangle \neq 0$ possible, even if O_m transforms under G in a nontrivial manner

order parameters: vacuum expectation values of operators that do not commute with the charges (VEVs of operators that transform nontrivially under G)

order parameters are quantitative measures of spontaneous symmetry breaking – if the vacuum is invariant under G, they vanish

proposition: the occurence of nonvanishing order parameters implies

$$\langle 0 | [Q_a, O_m(x)] | 0 \rangle \neq 0$$

for at least one charge Q_a and one operator O_m

proof: we assume the existence of a local operator $O_n(x)$, which transforms nontrivially under G, with $\langle 0 | O_n(x) | 0 \rangle \neq 0$

O_n belongs to a certain irreducible representation contoinded in $D(g)$ → we restrict our set of local operators to this irreducible representation:

$$v_m := \langle 0 | O_m(x) | 0 \rangle$$

$$\underbrace{\langle 0 | e^{i\alpha . Q} O_m(x) e^{-i\alpha . Q} | 0 \rangle = \underbrace{\left(e^{-i\alpha . T}\right)_{mn}}_{\substack{\text{irreducible} \\ \text{representation}}} \underbrace{\langle 0 | O_n(x) | 0 \rangle}_{v_n}}$$

$v \neq 0 \Rightarrow$ spans the whole representation space

$\Longrightarrow$ not all $T_a v$ vanish infinitesimal form:

$$i\alpha_a \langle 0 | [Q_a, O_m(x)] | 0 \rangle = -i\alpha_a (T_a)_{mn} v_n$$

$\Rightarrow$ not all $\langle 0 | [Q_a, O_m(x)] | 0 \rangle$ vanish,

i.e.

$$\langle 0 | [Q_a, O_m(x)] | 0 \rangle \neq 0$$

for at least one a and at least one m

$\Rightarrow$ at least one entry of the
<u>order parameter matrix</u>

$$C_{am} := -i \langle 0 | [Q_a, O_m(x)] | 0 \rangle$$

is different from zero

Goldstone theorem: Order parameters can take nonzero values only if the spectrum of the theory contains massless particles

remark 1: $G =$ Lie group is essential: SSB of discrete symmetries (like P, CP) does not give rise to Goldstone bosons

example:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi + \frac{\kappa}{2} \varphi^2 - \frac{\lambda}{4} \varphi^4 \, , \quad \lambda > 0$$

the discrete symmetry $\varphi \to -\varphi$ is spontaneously broken (for $\kappa > 0$) but there is no massless state (exercise)

remark 2: proof of Goldstone theorem given below makes use of Lorentz invariance; this is not essential, the Goldstone theorem theorem applies also to nonrelativistic systems; main difference: dispersion law not necessarily $\omega(\vec{k}) = |\vec{k}|$ $(v=c)$

examples: magnons of a ferromagnet $\omega \sim \vec{k}^2$

antiferromagnet $\omega \sim |\vec{k}|$

<u>remark 3</u>: $\mathcal{L}$ Lorentz invariant $\not\Rightarrow$ $|0\rangle$ Lorentz inv.

in principle, Lorentz invariance might break down spontaneously, just as well as an internal symmetry (usual situation in condensed matter physics)

particle physics: presently <u>no</u> experimental evidence that the ground state would single out a preferred frame of reference

$\rightarrow$ we restrict ourselves to theories with Lorentz invariant vacuum

$\Rightarrow \langle 0|J^\mu(x)|0\rangle = 0$, otherwise

$\langle 0|J^\mu(x)|0\rangle$ would single out a certain

direction in space-time

tensor fields: $\langle 0|T^{\mu\nu}(x)|0\rangle = \frac{g^{\mu\nu}}{4}\langle 0|T^\alpha{}_\alpha(x)|0\rangle$

reason: $T^{\mu\nu}$ does <u>not</u> transform irreducibly under the Lorentz group: $T^\alpha{}_\alpha$ represents an invariant subspace, transforming like a scalar field

the vacuum expectation value of the traceless part, $T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T^\alpha_\alpha$ vanishes

all of the local operators may be decomposed in this manner $\rightarrow$ only scalar or pseudo-scalar operators operators may give rise to order parameters

proof of the Goldstone theorem

nonvanishing order parameter $\Rightarrow$

$$\langle 0 | [Q, \mathcal{O}(x)] | 0 \rangle \neq 0$$

for at least one of the conserved charges and one of the local Lorentz scalars

$$F_\mu^+(x) := \langle 0 | J_\mu(x) \, \mathcal{O}(0) | 0 \rangle$$

$Q = \int d^3x \, J_0(x)$

Wightman function (<u>no</u> T symbol)

$\mathbb{1} = \sum_n |n\rangle \langle n|$ $\{|n\rangle\}$ complete set of energy-momentum eigenstates

$$\Rightarrow F_\mu^+(x) = \sum_n \langle 0 | \underbrace{J_\mu(x)}_{e^{iPx} J_\mu(0) e^{-iPx}} | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle$$

$$= \sum_n e^{-ip_n x} \langle 0 | J_\mu(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle$$

this expression contains only positive frequencies $p_n^0 \geq 0 \Rightarrow F_\mu^+(x)$ analytic in the lower half of the x^0 – plane (real x^0: $x^0 \to x^0 - i\varepsilon$)

Lorentz invariance $\Rightarrow F_\mu{}^+(x) = x_\mu F(x^2)$

$\partial^\mu J_\mu(x) = 0 \Rightarrow \partial^\mu F_\mu^+(x) = 0$

$$\Rightarrow \quad 4F(x^2) + x_\mu F'(x^2)\, 2x^\mu = 0$$

$$2F(x^2) + x^2 F'(x^2) = 0$$

ordinary first order differential equation

$$2F(u) + u\frac{dF(u)}{du} = 0 \quad \Big| \cdot \frac{du}{uF}$$

$$\frac{2\,du}{u} + \frac{dF}{F} = 0$$

$$2\ln u + \ln F = \text{const.}$$

$$F(u) \sim \frac{1}{u^2}$$

$$\Rightarrow \quad F_\mu^+(x) = \langle 0 | J_\mu(x)\, \mathcal{O}(0) | 0 \rangle$$

$$= -\frac{C}{2\pi^2} \frac{x_\mu}{x^4} = \frac{C}{4\pi^2} \partial_\mu \frac{1}{x^2}$$

remember: $\frac{1}{x^2} \rightarrow \frac{1}{(x^0 - i\varepsilon) - \vec{x}^2}$

scalar propagator $(m=0)$:

$$\Delta_0(x) = \Theta(x^0) \underbrace{i \int \frac{d^3p}{(2\pi)^3 2|\vec{p}|} e^{-ipx}}_{\Delta_0^+(x)}$$

$$+ \Theta(-x^0) \underbrace{i \int d\mu(p)\, e^{+ipx}}_{\Delta_0^-(x)}$$

$$\Delta_0^+(x) = i \int d\mu(p)\, e^{-ipx}$$

$$= \frac{i}{(2\pi)^3} \int d^4p\, \Theta(p^0)\, \delta(p^2)\, e^{-ipx}$$

$$\overset{\text{ex.}}{=} \frac{1}{4\pi^2 i \left[(x^0 - i\varepsilon)^2 - \vec{x}^2\right]}$$

analogously:

$$F_\mu^{-}(x) = \langle 0 | \mathcal{O}(0)\, J_\mu(x) | 0 \rangle = -\frac{C'}{2\pi^2}\frac{x_\mu}{x^4}$$

$$= \frac{C'}{4\pi^2}\partial_\mu \frac{1}{x^2} = i\,C'\,\partial_\mu \underbrace{\Delta_0^{+}(-x)}_{-\Delta_0^{-}(x)}$$

where $\frac{1}{x^2} \to \frac{1}{(x^0 + i\varepsilon)^2 - \vec{x}^2}$

causality $\Rightarrow$ $F_\mu^{+}(x) = F_\mu^{-}(x)$ for $x^2 < 0$ (spacelike)

$\Rightarrow C = C'$

$$\Rightarrow \langle 0 | [J_\mu(x), \mathcal{O}(0)] | 0 \rangle = F_\mu^{+}(x) - F_\mu^{-}(x)$$

$$= \frac{C}{4\pi^2}\partial_\mu \left[\underbrace{\frac{1}{(x^0 - i\varepsilon)^2 - \vec{x}^2}}_{P\frac{1}{x^2} + i\pi\varepsilon(x^0)\delta(x^2)} - \underbrace{\frac{1}{(x^0 + i\varepsilon)^2 - \vec{x}^2}}_{P\frac{1}{x^2} - i\pi\varepsilon(x^0)\delta(x^2)} \right]$$

$$= \frac{iC}{2\pi} \partial_\mu \left[\varepsilon(x^0)\, \delta(x^2) \right]$$

where $\varepsilon(x^0) = \begin{cases} +1 & x^0 > 0 \\ -1 & x^0 < 0 \end{cases}$

integrate $\mu = 0$ component over space at arbitrary $x^0 = t$

$$\delta(x^2) = \delta\left[(x^0)^2 - \vec{x}^{\,2} \right] =$$

$$= \delta\left[(x^0 - r)(x^0 + r) \right] \quad , \qquad r = |\vec{x}|$$

$$= \frac{1}{2r}\, \delta(x^0 - r) + \frac{1}{2r}\, \delta(x^0 + r)$$

$$\Rightarrow \int d^3x \, \langle 0 | \left[J_0(t, \vec{x}), \sigma(0) \right] | 0 \rangle$$

$$= \frac{iC}{2\pi} \int d^3x \, \frac{\partial}{\partial x^0} \left[\varepsilon(x^0)\, \delta(x^2) \right] \Big|_{x^0 = t}$$

$$= \frac{iC}{2\pi} \frac{\partial}{\partial x^0} \varepsilon(x^0)\, 4\pi \int_0^\infty \frac{dr\, r^2}{2r} \left[\delta(x^0 - r) + \delta(x^0 + r)\right] \Big|_{x^0 = t}$$

$$= i\, C \frac{\partial}{\partial x^0} \varepsilon(x^0) \int_{-\infty}^{+\infty} dr\, r\, \Theta(r) \left[\delta(x^0 - r) + \delta(x^0 + r)\right] \Big|_{x^0 = t}$$

$$= i\, C \frac{\partial}{\partial x^0} \varepsilon(x^0) \underbrace{\left[x^0\, \Theta(x^0) - x^0\, \Theta(-x^0)\right]}_{\varepsilon(x^0)\, x^0} \Big|_{x^0 = t}$$

$$= i\, C \frac{\partial}{\partial x^0} x^0 = i\, C$$

$$\Rightarrow \quad \underbrace{\langle 0 | [Q, \mathcal{O}(0)] | 0 \rangle}_{\neq 0} = i\, C$$

$$\Rightarrow \quad C \neq 0$$

occurence of the massless propogator in the above explicit expression for the Wightman function signals the presence of a massless particle :

$$F_\mu^+(x) = \langle 0 | J_\mu(x)\, \mathcal{O}(0) | 0 \rangle$$

$$= \sum_n e^{-ip_n x} \langle 0 | J_\mu(0) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle$$

$$= i\, C\, \partial_\mu \Delta_0^+(x)$$

$$= \frac{iC}{(2\pi)^3} \int d^4p\ \Theta(p^0)\, \delta(p^2)\, p_\mu\, e^{-ipx}$$

$$= i\, C \underbrace{\int d\mu(p)}_{\frac{d^3p}{(2\pi)^3 2p^0}} p_\mu\, e^{-ipx} \quad , \quad p^0 = |\vec{p}|$$

<u>only</u> intermediate states with <u>$p_n^2 = 0$</u> contribute

essential point: $C \neq 0 \Rightarrow$ spectrum contains massless particles

<u>remark:</u> $\mathcal{O}$ scalar $\Rightarrow$ $\langle n | \mathcal{O}(0) | 0 \rangle \neq 0$ only if angular momentum of $|n\rangle$ vanishes $\Rightarrow \langle 0 | J^\mu(0) | n \rangle \sim p_n^\mu$; current conservation $\Rightarrow p_n^\mu \langle 0 | J_\mu(0) | n \rangle = 0$; both conditions can only be met if $p_n^2 = 0$

<u>number of Goldstone bosons</u>

so far, our proof showed that the spectrum of the theory contains one or more Goldstone bosons

denote the corresponding one-particle states by $|\pi_r(p)\rangle$, $r = 1, \dots N_\pi$

we want to determine the <u>minimal number</u> of massless particles required by the spontaneous breakdown of a continuous symmetry

# of conserved currents J^μ_a = dim (G) =
= # of the generators of the Lie group G =
= dimension of the <u>Lie algebra</u>

$$\mathcal{L} = \left\{ Q \mid Q = \sum_a \alpha_a Q_a \, , \, \alpha_a \in \mathbb{R} \right\}$$

$$\underbrace{\langle 0| e^{i\alpha\cdot Q} O_m e^{-i\alpha\cdot Q} |0\rangle}_{\langle 0|O_m|0\rangle + i\alpha_a \langle 0|[Q_a, O_m]|0\rangle + \mathcal{O}(\alpha^2)} = \underbrace{\left(e^{-i\alpha\cdot T}\right)_{mn} \langle 0|O_n|0\rangle}_{\langle 0|O_m|0\rangle - i\alpha_a (T_a)_{mn} \langle 0|O_n|0\rangle + \mathcal{O}(\alpha^2)}$$

we consider those elements of $\mathcal{L}$ for which

$$\langle 0| e^{i\alpha\cdot Q} O_m e^{-i\alpha\cdot Q} |0\rangle = \langle 0| O_m |0\rangle$$

<u>$\forall$ local operators</u> O_m

this is equivalent with the condition

$$\langle 0| [Q, O_m] |0\rangle = 0 \quad \forall \; O_m$$

$$\mathcal{H} := \{ Q \in \mathcal{L} \mid \langle 0| [Q, O_m] |0\rangle = 0 \;\; \forall O_m \}$$

is a <u>subalgebra</u> of $\mathcal{L}$, which means

that $Q, Q' \in \mathcal{H} \Rightarrow \alpha Q + \alpha' Q' \in \mathcal{H}$

and $[Q, Q'] \in \mathcal{H}$

(<u>proof</u> is left as a simple home work problem)

$\Rightarrow$ $\mathcal{H}$ generates a <u>subgroup</u> $H \subset G$

$\Rightarrow$ decomposition of Lie algebra $\mathcal{L} = \mathcal{H} + \mathcal{K}$

$\mathcal{H}$ is spanned by those charges that annihilate the ground state (they represent symmetries of the vacuum)

$\mathcal{K}$ exclusively contains elements with "$Q|0\rangle \neq 0$"

$\mathcal{K}$ does <u>not</u> form a subalgebra!

SSB $\Leftrightarrow$ $\dim(H) < \dim(G)$

$Q \in \mathcal{K} \Rightarrow \langle 0| [Q, O_m] |0\rangle \neq 0$ for at least one Lorentz scalar O_m

we choose a basis of $\mathcal{L}$ where

$H_1, \dots, H_{\dim(H)}$ spans the subalgebra $\mathcal{H}$, and

$K_1, \dots, K_{\dim(G) - \dim(H)}$ spans $\mathcal{K}$

$$\mathcal{H} = \langle H_i \rangle_{i=1}^{\dim(H)} \, , \quad \mathcal{K} = \langle K_s \rangle_{s=1}^{\dim(G)-\dim(H)}$$

<u>order parameter matrix</u>

$$C_{am} = -i \langle 0 | [Q_a, O_m] | 0 \rangle$$

has two different types of matrix elements with respect to our new basis:

$$C_{im} = -i \langle 0 | [H_i, O_m] | 0 \rangle = 0 \quad \forall m$$

$$C_{sm} = -i \langle 0 | [K_s, O_m] | 0 \rangle$$

↑

for given s different from zero
for at least one index $m = m(s)$

furthermore, a linear combination $\sum_s \alpha_s K_s$ with $\sum_s \alpha_s C_{sm} = 0 \quad \forall m$ is excluded, as this would imply $\sum_s \alpha_s K_s \in \mathcal{H}$

form of the order parameter matrix
($C_{im} \equiv 0$ not shown)

$$
\left.
\begin{array}{lcccc}
 & O_1 & O_2 & \cdots & \\
K_1 & \blacksquare & & & \\
K_2 & & \blacksquare & & \\
\vdots & & & \ddots & \\
K_{d(G)-d(H)} & & & & \blacksquare
\end{array}
\right\} \begin{array}{l} d(G) - d(H) \\ \underline{\text{linearly independent}} \\ \text{rows} \end{array}
$$

$\Rightarrow$ the order parameter matrix C_{am} contains $d(G) - d(H)$ linearly independent rows

order parameter C_{sm} determines the Wightman function

$$
\langle 0 | J_5^{\mu}(x)\, O_m(0) | 0 \rangle = i\, C_{sm} \int \frac{d^3p}{(2\pi)^3\, 2p^0}\, p^{\mu} e^{-ipx}
$$

$$
p^0 = |\vec{p}\,|
$$

$$\mathbb{1} = \sum_n |n\rangle\langle n| = \sum_{r=1}^{N_\pi} \int \frac{d^3p}{(2\pi)^3 2p^0} \underbrace{|\pi_r(p)\rangle\langle\pi_r(p)|}_{\text{massless scalar states}} + \dots$$

$$\Rightarrow \langle 0| J_s^\mu(x)\, O_m(0)|0\rangle \overset{\text{only massless scalar states contribute}}{\doteq}$$

$$= \sum_{r=1}^{N_\pi} \int d\mu(p) \underbrace{\langle 0| J_s^\mu(x)|\pi_r(p)\rangle}_{i F_{sr} p^\mu e^{-ipx}} \underbrace{\langle \pi_r(p)| O_m(0)|0\rangle}_{G_{rm}}$$

$$\Rightarrow C_{sm} = \sum_{r=1}^{N_\pi} F_{sr} G_{rm}$$

C_{sm} contains $\dim(G) - \dim(H)$ linearly independent rows $\Rightarrow N_\pi \geq \dim(G) - \dim(H)$

<u>remark:</u> the above argument leads to an <u>inequality</u>; reason: symmetry arguments alone do <u>not</u> suffice to exclude the occurence of additional massless particles;

<u>trivial example</u>: imagine that we had been looking at a subgroup of the full symmetry group without noticing and that some of the extra generators do not annihilate the vacuum → entire discussion would have gone through, but we would have missed some of the Goldstone bosons

<u>example for SSB: vacuum expectation value of a sclar field</u>

order parameter $\langle 0|\phi(x)|0\rangle \neq 0$ for scalar field $\phi(x)$

we consider the following <u>model</u>:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi - V(\phi)$$

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix}$$ N <u>real</u> scalar fields

$\mathcal{L}$ invariant under Lie group G with representation $e^{-i\alpha_a T_a}$ acting on ϕ (restricts potential $V(\phi)$)

<u>remark:</u> $T_a{}^T = -T_a = T_a{}^*$ (ϕ real)

Noether current: $J_a^\mu = \frac{i}{2} \phi^T T_a \overleftrightarrow{\partial^\mu} \phi$

$$Q_a = \int d^3x \, J_a^0$$

infinitesimal form of $e^{i\alpha.Q} \phi \, e^{-i\alpha.Q} = e^{-i\alpha.T} \phi$:

$$[Q_a, \phi_m] = -(T_a)_{mn} \phi_n$$

we assume: $\langle 0 | \phi_m(x) | 0 \rangle = v_m$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \neq 0$$

$$\underbrace{\langle 0 | [Q_a, \phi_m] | 0 \rangle}_{i \, C_{am}} = -(T_a)_{mn} v_n$$

explicit determination of the Goldstone bosons of our scalar model (tree approximation)

a) vacuum expectation value (in the tree approximation) determined by the minimum of $V(\phi)$:

$$\left.\frac{\partial V}{\partial \phi_m}\right|_{\phi = v} = 0 \quad \forall m$$

b) mass matrix

$$V(\phi) = V(v) + \frac{1}{2!} \left.\frac{\partial^2 V}{\partial \phi_m \partial \phi_n}\right|_{\phi = v} (\phi - v)_m (\phi - v)_n + \ldots$$

$$\phi' := \phi - v$$

$\rightarrow$ mass matrix $\quad (M^2)_{mn} = \left.\dfrac{\partial^2 V}{\partial \phi_m \partial \phi_n}\right|_{\phi = v}$

c) invariance of $V(\phi)$

$$V(\phi) = V(e^{-i\alpha . T}\phi)$$

$$= V(\phi) - i\,\alpha_a\,(T_a\phi)_n \frac{\partial V}{\partial \phi_n} + \dots$$

$$\Rightarrow \quad \frac{\partial V}{\partial \phi_n}(T_a\phi)_n = 0$$

d) massless degrees of freedom

differentiate previous equation $\left(\frac{\partial}{\partial \phi_m}\dots\right)$

$$\rightarrow \quad \frac{\partial^2 V}{\partial \phi_m \partial \phi_n}(T_a\phi)_n + \frac{\partial V}{\partial \phi_n}(T_a)_{nm} = 0$$

insert $\phi = v \rightarrow \quad (M^2)_{mn}(T_a v)_n = 0 \ \forall a$

e) number of Goldstone bosons

dimension of space spanned by $\{T_a v\}_a$
= # of Goldstone bosons = "# of generators with $T_a v \neq 0$"

$\{b_1, \dots, b_{N_\pi}\}$ ONB of $\langle\langle \{T_a v\}_a \rangle\rangle$

$\Rightarrow \quad \pi_r = b_r^T \phi'$, $\quad r = 1, \dots, N_\pi = \dim(G) - \dim(H)$

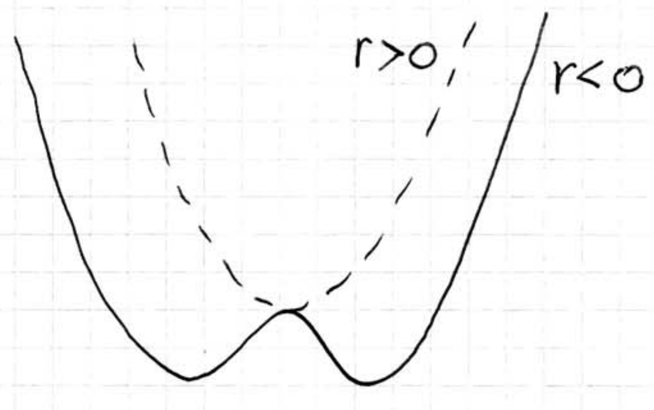
f) unbroken subgroup H generated by those generators $T = \sum_a \alpha_a T_a$ with $Tv = 0$

<u>example:</u> $G = SO(N)$, $N \geq 2$

$$\dim(G) = \frac{N(N-1)}{2} = \binom{N}{2}$$

ϕ transforming according N-dim. (defining) repr. of $SO(N)$ (<u>linear σ-model</u>)

$$V(\phi) = \frac{1}{2} r \phi^T \phi + \frac{\lambda}{4} (\phi^T \phi)^2$$



$r > 0$ <u>no</u> SSB

$\rightarrow$ N scalars with $M^2 = r$

<u>$r < 0$ SSB</u>

$$\|\phi\| = \sqrt{\phi^T \phi}$$

$$V = \frac{r}{2} \|\phi\|^2 + \frac{\lambda}{4} \|\phi\|^4$$

$$\frac{\partial V}{\partial \|\phi\|} = r \|\phi\| + \lambda \|\phi\|^3 = \|\phi\| \left(r + \lambda \|\phi\|^2\right) = 0$$

$r < 0 \quad \rightarrow \quad$ minimum at $|\phi\|^2 = -\frac{r}{\lambda}$

choose $\quad v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \|v\| \end{bmatrix} \quad , \quad \|v\| = \sqrt{-\frac{r}{\lambda}}$

$H = SO(N-1)$ unbroken subgroup

$$\Rightarrow N_\pi = \dim(G) - \dim(H) =$$

$$= \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2}$$

$$= N - 1 \quad \text{Goldstone bosons}$$

$\Rightarrow$ $\exists$ N-1 generators which do <u>not</u> annihilate v :

$$iT_1 = \left[\begin{array}{c|c} \bigcirc & \begin{array}{c}1\\0\\ \vdots \\ 0\end{array} \\ \hline -1\,0 \ldots 0 & 0 \end{array}\right], \quad iT_2 = \left[\begin{array}{c|c} \bigcirc & \begin{array}{c}0\\1\\0\\ \vdots \\ 0\end{array} \\ \hline 0\,1\,0 \ldots 0 & 0 \end{array}\right], \ldots$$

$$\langle\!\langle \{ iT_a v \mid a = 1, \ldots, N-1 \} \rangle\!\rangle =$$

$$= \langle\!\langle \{ e_1, \ldots, e_{N-1} \} \rangle\!\rangle$$

$\pi_1 = \phi_1 , \ldots , \pi_{N-1} = \phi_{N-1}$ massless Goldstone fields

only $\phi_N' = \phi_N - \|v\|$ is massive

check by inspection of $V(\phi)$:

$$V(\phi) = V(v + \phi') =$$

$$= \frac{1}{2} r \left[\phi_1^2 + \ldots + \phi_{N-1}^2 + (\phi_N' + \|v\|)^2 \right]$$

$$+ \frac{\lambda}{4} \left[\qquad - \!\!/\!/\!\! - \qquad \right]^2 =$$

$$= \frac{r}{2}\left[\phi_1^2 + \ldots + \phi_{N-1}^2 + (\phi_N' + \|v\|)^2\right]$$

$$+ \frac{\lambda}{4}\left[\phi_1^2 + \ldots + \phi_{N-1}^2\right]^2$$

$$+ \frac{\lambda}{2}\left[\phi_1^2 + \ldots + \phi_{N-1}^2\right](\phi_N' + \|v\|)^2$$

$$+ \frac{\lambda}{4}(\phi_N' + \|v\|)^4$$

$$= \underbrace{\frac{r}{2}\|v\|^2 + \frac{\lambda}{4}\|v\|^4}_{V(v) = \text{const.}}$$

$$+ \underbrace{(r\|v\| + \lambda\|v\|^3)}_{0}\,\phi_N'$$

$$+ \frac{1}{2}\underbrace{(r + \lambda\|v\|^2)}_{0}(\phi_1^2 + \ldots + \phi_{N-1}^2)$$

$$+ \frac{1}{2}\underbrace{(r + 3\lambda\|v\|^2)}_{2\lambda\|v\|^2 = M_N^2}(\phi_N')^2$$

+ *cubic and quartic interaction terms*