16. Quantization of nonabelian gauge fields

$$\mathcal{L}[A] = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$$

simple gauge group G with structure constants f_{abc}

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) + \mathcal{L}_{int}$$

$$S_0 = -\frac{1}{4} \int d^4x \, (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$$

$$= \int d^4x \, d^4y \, \frac{1}{2} A^a_\mu(x) K^{\mu\nu}(x,y) A^a_\nu(y)$$

$$K^{\mu\nu}(x,y) = (g^{\mu\nu} \Box - \partial^\mu \partial^\nu) \delta^{(4)}(x-y)$$

K is not <u>invertible</u> $\to$ propagator cannot be defined without <u>gauge condition</u>

$\to$ quantization with constraint $\to$ trick of <u>Faddeev</u> and <u>Popov</u>

<u>observation:</u> gauge field A_μ given $\to$ observable quantities <u>unchanged</u> under gauge transformation

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1}$$

where $U(\alpha(x)) = e^{-i\alpha_a(x) T_a}$

$\mathcal{L}[A] = \mathcal{L}[A'] \Rightarrow$ action S constant on "gauge orbit"

$\rightarrow$ divergence of functional integral

integration over gauge group (at all space-time points) should be pulled out of functional integral

$$\int \prod_x d\mu(g(x)) \quad \ldots$$

invariant group measure $d\mu(g)$, $g \in G$

$$d\mu(g'g) = d\mu(g)$$

<u>example</u>: $SU(2)$

group element $g \in SU(2)$ parametrized by Euler angles $0 \le \alpha \le 2\pi$, $0 \le \beta \le \pi$, $0 \le \gamma \le 4\pi$

$$g(\alpha, \beta, \gamma) = \exp\left(i\alpha \frac{\sigma_3}{2}\right) \exp\left(i\beta \frac{\sigma_2}{2}\right) \exp\left(i\gamma \frac{\sigma_3}{2}\right)$$

$$\int d\mu(g) \ldots = \frac{1}{16\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{4\pi} d\gamma \sin\gamma \ldots$$

<u>gauge condition</u> $f_a[A(x)] = \omega_a(x)$

<u>assumption</u>: for given $A(x), \omega(x)$ $\exists!\, g(x) \equiv (\alpha_1(x), \ldots, \alpha_n(x))$

with $f_a[A'(x)] = \omega_a(x)$, $A' \equiv A^g$

$$1 = \underbrace{\Delta_f[A]}_{\substack{\text{Faddeev - Popov}\\ \text{determinant}}} \int \prod_x d\mu(g(x)) \prod_a \delta[f_a[A^g(x)] - \omega_a(x)]$$

$\Delta_f[A]$ is gauge invariant:

$$\Delta_f[A^{g'}]^{-1} = \int [d\mu(g)]\, \delta[f[A^{g'g}] - \omega]$$

$$= \int [d\mu(g'g)]\, \delta[f[A^{g'g}] - \omega]$$

$$= \Delta_f[A]^{-1}$$ (invariant group measure)

$$\langle 0 | T \overbrace{O[A]}^{\text{gauge invariant}} | 0 \rangle =$$

$$= \frac{1}{\mathcal{N}} \int [dA]\, \Delta_f[A] \int [d\mu(g)]\, \delta[f[A^g] - \omega]\, e^{iS[A]}\, O[A]$$

gauge transformation $A \to A^{g^{-1}}$

$$[dA^g] = [dA]$$

$$A_\mu \to A'_\mu = U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1}$$

$$\Rightarrow \quad dA_\mu \to U \, dA_\mu U^{-1}$$

$$\Rightarrow \quad ds^2 = \int d^4x \; \mathrm{Tr} (dA_\mu \, dA^\mu) \quad \text{is gauge inv.}$$

$$\Rightarrow \quad [dA] \text{ is gauge invariant}$$

$$\Rightarrow \quad \langle 0 | T \, \mathcal{O}[A] \, | 0 \rangle =$$

$$= \frac{1}{\mathcal{N}} \int [dA^{g^{-1}}] \, \Delta_f [A^{g^{-1}}] \int [d\mu(g)] \, \delta [f[A] - \omega] \times e^{iS[A^{g^{-1}}]} \mathcal{O}[A^{g^{-1}}]$$

$$= \frac{1}{\mathcal{N}} \int [d\mu(g)] \underbrace{\int [dA] \, \Delta_f [A] \; \delta [f[A] - \omega] \, e^{iS[A]} \mathcal{O}[A]}_{\text{independent of } g}$$

$$= \frac{\int [dA] \, \Delta_f [A] \, \delta [f[A] - \omega] \, e^{iS[A]} \mathcal{O}[A]}{\int [dA] \, \Delta_f [A] \, \delta [f[A] - \omega] \, e^{iS[A]}}$$

computation of $\Delta_f[A]$

multiplied by $\delta[f[A]-\omega]$ → need $\Delta_f[A]$ only for those A which satisfy $f_a[A]=\omega_a$

$$\Delta_f[A]^{-1} = \int \prod_x d\mu(g(x)) \prod_a \delta[f_a[A^{g(x)}]-\omega_a(x)]$$

analogous to $\delta[f(x)] = \frac{1}{|f'(x_0)|}\delta(x-x_0)$,

$$f(x_0)=0$$

integration over g: zero for $g=e$ → only small neighbourhood of e relevant → infinitesimal g → group measure $d\mu(g) \to d\alpha_1 \ldots d\alpha_n$ (infinitesimal transformation is abelian)

$$A'_\mu = e^{-i\alpha.T} A_\mu e^{i\alpha.T} + \frac{i}{g}\left(\partial_\mu e^{-i\alpha.T}\right) e^{i\alpha.T}$$

$$= A_\mu - i[\alpha.T, A_\mu] + \frac{i}{g}(-i\partial_\mu \alpha.T) + \mathcal{O}(\alpha^2)$$

$$= A_\mu^a T_a - i\alpha_b A_\mu^c \underbrace{[T_b, T_c]}_{if_{bca}T_a} + \frac{1}{g}\partial_\mu \alpha_a T_a + \mathcal{O}(\alpha^2)$$

$$= \left(A_\mu^a + \alpha_b A_\mu^c f_{bca} + \frac{1}{g} \partial_\mu \alpha_a \right) T_a + O(\alpha^2)$$

$$\Rightarrow A_\mu^{\prime a} - A_\mu^a = \frac{1}{g} \underbrace{\left(\delta_{ab} \partial_\mu + g A_\mu^c f_{cab} \right)}_{D_{\mu, ab}} \alpha_b + O(\alpha^2)$$

$$D_{\mu, ab} = \delta_{ab} \partial_\mu + i g A_\mu^c \underbrace{(-i f_{cab})}_{(t_c)_{ab}}$$

covariant derivative (adjoint representation)

$$f_a[A^g] - \omega_a = f_a[A] - \omega_a + \frac{\delta f_a[A]}{\delta A_\mu^c} (A_\mu^{\prime c} - A_\mu^c) + \dots$$

$$= \frac{1}{g} \frac{\delta f_a[A]}{\delta A_\mu^c} D_{\mu, cb} \alpha_b + \dots$$

$$= \frac{1}{g} M_{ab}^f \alpha_b + \dots$$

corresponding integral kernel:

$$M^f_{ab}(x,y) = \int d^4z \, \frac{\delta f_a[A(x)]}{\delta A^c_\mu(z)} D_{\mu,cb}(z,y)$$

$$D_{\mu,ab}(x,y) = \left[\delta_{ab} \frac{\partial}{\partial x^\mu} + g f_{cab} A^c_\mu(x) \right] \delta^{(4)}(x-y)$$

$$\Rightarrow \Delta_f[A]^{-1} = \int \prod_x d\alpha_1(x) \dots d\alpha_n(x) \prod_a \delta\left(\frac{1}{g} \int d^4y \, M^f_{ab}(x,y) \alpha_b(y) \right)$$

$$\equiv \int \underbrace{[d\alpha_1 \dots d\alpha_n]}_{[d\alpha]} \delta\left(\frac{1}{g} M^f \alpha \right)$$

$$= \int [d\alpha] \frac{1}{\det(M^f/g)} \delta(\alpha)$$

$$= \frac{1}{\det(M^f/g)}$$

$$\Rightarrow \Delta_f[A] = \det(M^f/g)$$

using Grassmann variables $\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n$,
we can write

$$\det B = \int d\xi_1 \dots d\xi_n \, d\eta_1, \dots d\eta_n \, e^{\eta_a B_{ab} \xi_b}$$

→ functional integral representation of $\det M^f$:

$$\det M^f \sim \int \prod_a [d\xi_a \, d\eta_a] \, e^{-i \int d^4x \, d^4y \, \eta_b(x) \, M^f_{bc}(x,y) \, \xi_c(y)}$$

usual notation: $\xi_a = c_a$, $\eta_a = \bar{c}_a$

$\Rightarrow \langle 0 | T \, \mathcal{O}[A] \, | 0 \rangle =$

$$= \frac{1}{\mathcal{N}} \int [dA] \, \delta[f[A] - \omega] \int [dc \, d\bar{c}] \, e^{i \{ S[A] + S_{FP}[A, c, \bar{c}] \}} \mathcal{O}[A]$$

$$S_{FP}[A, c, \bar{c}] = -\int d^4x \, d^4y \, \bar{c}_a(x) \, M^f_{ab}(x,y) \, c_b(y)$$

$c(x)$: ghost field (Faddeev - Popov ghost)
scalar field with "wrong" statistics (anticommuting)
(factor -1 for ghost loops)

technical problem: integration over δ-function

<u>observations:</u>

(i) $\langle 0|T\,\mathcal{O}[A]|0\rangle$ independent of the choice of $\omega_a(x)$

(ii) S_{FP} independent of $\omega_a(x)$

$\rightarrow$ integrate over ω with weight function $F[\omega]$

usual choice: $F[\omega] = e^{-\frac{i\xi}{2}\int d^4x\, \omega_a(x)\,\omega_a(x)}$

$$\rightarrow \langle 0|T\,\mathcal{O}[A]|0\rangle =$$

$$= \frac{1}{\mathcal{N}} \int [dA\, dc\, d\bar{c}]\; e^{i S_{eff}[A,c,\bar{c}]}\; \mathcal{O}[A]$$

$$S_{eff} = S[A] + S_{GF}[A] + S_{FP}[A,c,\bar{c}]$$

gauge fixing $S_{GF}[A] = -\frac{\xi}{2}\int d^4x\, f_a[A(x)]\, f_a[A(x)]$

R_ξ gauge : $\quad f_a[A] = \partial_\mu A^\mu_a$

covariant gauge

terms $\sim A^2$:

$$-\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu\right)$$

$$-\frac{\xi}{2}\,\partial^\mu A_\mu^a\;\partial^\nu A_\nu^a$$

partial int. in S

$$\rightarrow \qquad -\frac{1}{2} A_\mu^a\left[-g^{\mu\nu}\,\Box + (1-\xi)\,\partial^\mu\partial^\nu\right] A_\nu^a$$

$\rightarrow$ gauge boson propagator in R_ξ gauge :

$$-\frac{i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{k_\mu k_\nu}{k^2}\right]$$

$\xi = 1$: 't Hooft – Feynman gauge $\quad \dfrac{-i\,g_{\mu\nu}}{k^2+i\varepsilon}$

$\xi \to \infty$: Landau gauge $\quad \dfrac{-i}{k^2+i\varepsilon}\left(g_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2}\right)$

16/11

$$S_{FP} = -\int d^4x\, d^4y\ \bar{c}_a(x)\, M^f_{ab}(x,y)\, c_b(y)$$

$$M^f_{ab}(x,y) = \int d^4z\ \frac{\delta f_a[A(x)]}{\partial A^c_\mu(z)}\ D_{\mu,cb}(z,y)$$

in our case: $\dfrac{\delta f_a[A(x)]}{\delta A^c_\mu(z)} = \dfrac{\delta}{\delta A^c_\mu(z)}\ \partial^\nu_x A^a_\nu(x)$

$$= \partial^\nu_x\, \delta^{(4)}(x-z)\, \delta_{ac}\, g^\mu_\nu = \delta_{ac}\, \partial^\mu_x\, \delta^{(4)}(x-z)$$

$$\Rightarrow M^f_{ab}(x,y) = \partial^\mu_x\, D_{\mu,ab}(x,y) =$$

$$= \frac{\partial}{\partial x_\mu}\left[\delta_{ab}\frac{\partial}{\partial x^\mu} + g\, f_{cab}\, A^c_\mu(y)\right]\delta^{(4)}(x-y)$$

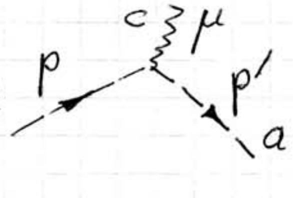
$$= \left[\delta_{ab}\, \Box_x + g\, f_{cab}\, A^c_\mu(y)\ \partial^\mu_x\right]\delta^{(4)}(x-y)$$

$$S_{FP} = -\int d^4x\, d^4y\ \bar{c}_a(x)\left[\delta_{ab}\,\Box_x + g\, f_{cab}\, A^c_\mu(y)\ \partial^\mu_x\right]\delta^{(4)}(x-y)\, c_b(y)$$

$$= \int d^4x\ \left[\partial_\mu \bar{c}_a(x)\, \partial^\mu c_a(x) + \left(\partial^\mu \bar{c}_a(x)\right) g\, f_{abc}\, c_b(x)\, A^c_\mu(x)\right]$$

ghost propagator $\dfrac{i\,\delta_{ab}}{k^2 + i\varepsilon}$

vertex



$-g f_{abc}\, p'_\mu$

Feynman rules of a nonabelian gauge theory with spin 1/2 fermions in the R_ξ gauge

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_{GF} + \mathcal{L}_{FP} + \mathcal{L}_\psi$$

$$\mathcal{L}_\psi = \overline{\psi}\,(i\,\not{D} - m)\,\psi$$

$$\overline{\psi}\,\not{D}\,\psi = \overline{\psi}_i\,\gamma^\mu (D_\mu)_{ij}\,\psi_j$$

$$D_\mu = \partial_\mu + i g\, T_a A^a_\mu$$

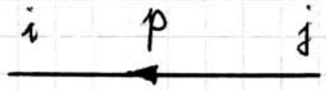
$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}{}^a F_a{}^{\mu\nu}, \quad F_{\mu\nu}{}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$$

$$\mathcal{L}_A + \mathcal{L}_{GF} = \mathcal{L}_{kin} + \mathcal{L}_{int}[A]$$

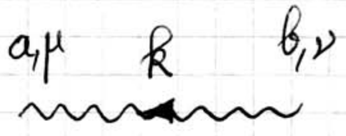
$$\mathcal{L}_{int}[A] = \frac{g}{2}\,(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\, f_{abc}\, A^\mu_b A^\nu_c$$

$$- \frac{g^2}{4} f_{abc} f_{ade} A^b_\mu A^c_\nu A^\mu_d A^\nu_e$$

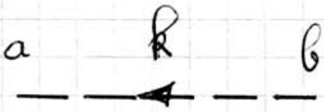
$$= g f_{abc}\, \partial_\mu A^a_\nu\, A^\mu_b A^\nu_c - \frac{g^2}{4} f_{abc} f_{ade} A^b_\mu A^c_\nu A^\mu_d A^\nu_e$$



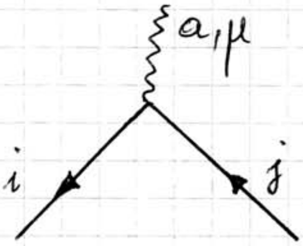
$$\frac{1}{i}\,\frac{\delta_{ij}}{m-\not p-i\varepsilon}$$



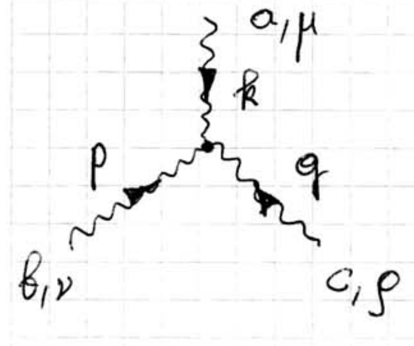
$$-\frac{i\,\delta_{ab}}{k^2+i\varepsilon}\left[g_{\mu\nu}-\left(1-\frac{1}{\xi}\right)\frac{k_\mu k_\nu}{k^2}\right]$$



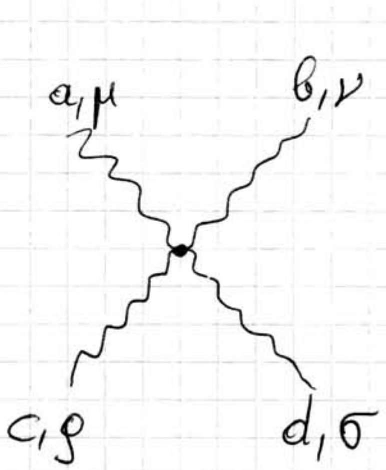
$$\frac{i\,\delta_{ab}}{k^2+i\varepsilon}$$



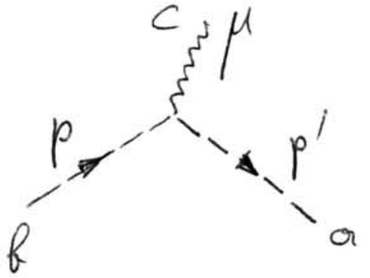
$$-i\,g\,(T_a)_{ij}$$



$$\begin{aligned}-g\,f_{abc}\,\big[\,&g_{\mu\nu}\,(k-p)_\rho\;+\\ &+g_{\nu\rho}\,(p-q)_\mu\\ &+g_{\rho\mu}\,(q-k)_\nu\,\big]\end{aligned}$$



$$\begin{aligned}-i g^2\big[&f_{abe}f_{cde}\,(g_{\mu\rho}\,g_{\nu\sigma}-g_{\mu\sigma}\,g_{\nu\rho})\\ &+f_{ace}f_{bde}\,(g_{\mu\nu}\,g_{\rho\sigma}-g_{\mu\sigma}\,g_{\nu\rho})\\ &+f_{ade}f_{bce}\,(g_{\mu\nu}\,g_{\rho\sigma}-g_{\mu\rho}\,g_{\nu\sigma})\big]\end{aligned}$$



$$-g f_{abc}\, p'_\mu$$