15. Nonabelian gauge theories

reminder: QED is abelian gauge theory

(gauge group $U(1)$)

QED of spin ½ fields $\psi(x)$ and complex scalar fields $\phi(x)$

e.g. $\psi = \begin{bmatrix} e \\ \mu \\ \tau \\ u_r \\ u_g \\ u_b \\ \vdots \end{bmatrix}$, $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{bmatrix}$

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i \gamma^\mu \partial_\mu - m \right) \psi$$

(m: mass matrix)

$$+ (D_\mu \phi)^\dagger D^\mu \phi - V(\phi)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu \psi = (\partial_\mu + ie Q A_\mu) \psi$$

(Q: charge matrix of spin ½ fields)

$$D_\mu \phi = (\partial_\mu + ie Q' A_\mu) \phi$$

(Q': charge matrix of spin 0 fields)

$\mathcal{L}_{QED}$ is invariant under the local $U(1)$ gauge transformation

$$\psi(x) \to e^{-iQ\alpha(x)} \psi(x)$$

$$\phi(x) \to e^{-iQ'\alpha(x)} \phi(x)$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

- generalization: nonabelian gauge group (Yang - Mills 1954)

compact Lie group G

$$U(\alpha_1, \dots, \alpha_n) = e^{-i\alpha_a T_a} \qquad \alpha_a \in \mathbb{R}$$

$T_a^\dagger = T_a$, $a = 1, \dots, n =: d(G)$ = number of generators

Lie algebra $\mathcal{L}$ (of G) spanned by all real linear combinations of $T_1, \dots, T_n$

commutation relation

$$[T_a, T_b] = i f_{abc} T_c$$

(f_{abc}: structure constants)

usual choice: orthogonal basis $\mathrm{Tr}(T_a T_b) = c\,\delta_{ab}$

$\Rightarrow$ f_{abc} totally antisymmetric (exercise)

(linear) representation of Lie algebra $\mathcal{L}$:

$T_a \to D(T_a)$, $D(T_a)$ are linear operators with $[D(T_a), D(T_b)] = i f_{abc} D(T_c)$

adjoint representation: the matrices $(t_a)_{bc} = -i f_{abc}$ are building an n-dimensional representation of the Lie algebra $\mathcal{L}$ (exercise)

examples:

1) $SU(2)$ $\quad U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}$

three real parameters $\alpha_1, \alpha_2, \alpha_3$

Pauli matrices $\sigma_1, \sigma_2, \sigma_3$

$T_a = \frac{\sigma_a}{2}$ generators of $SU(2)$

Lie algebra $su(2)$ of $SU(2)$ characterized

by fundamental commutation relations

$$[T_a, T_b] = i \varepsilon_{abc} T_c$$

with structure constants $\varepsilon_{abc} = \begin{cases} +1 & a,b,c \text{ even perm. of } 1,2,3 \\ -1 & \text{-"- odd -"- -"-} \\ 0 & \text{otherwise} \end{cases}$

of $su(2)$

normalization: $\mathrm{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$

adjoint representation: three 3×3 matrices $(t_a)_{bc} = -i \varepsilon_{abc}$

$$[t_a, t_b] = i \varepsilon_{abc} t_c$$

2) $SU(3)$ = group of unitary 3×3 matrices U with $\det U = 1$

Lie algebra $su(3) = \{ T \mid T^\dagger = T \wedge \mathrm{Tr}\, T = 0 \}$

$\Rightarrow \dim su(3) = 8 = d(SU(3))$

defining representation of $SU(3)$:

basis $T_a = \frac{\lambda_a}{2}$

Gell-Mann matrices $\lambda_1, \dots, \lambda_8$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

<u>construction of a nonabelian gauge theory</u>

$$\mathcal{L}_\psi = i \bar{\psi} \gamma^\mu (\partial_\mu + i g T_a A_\mu^a) \psi - \bar{\psi} m \psi$$

ψ fermionic multiplet with respect to gauge

group G with representation $U = e^{-i\alpha_a T_a}$

$[T_a, m] = 0$ restricts mass matrix m

analogously:

$$\mathcal{L}_\phi = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi)$$

for scalar multiplet ϕ

$$D_\mu \phi = (\partial_\mu + i g T_a' A_\mu^a)\phi$$

<u>example: QCD</u> $G = SU(3)$

each quark flavour (u, d, c, s, t, b) has three "colour" degrees of freedom, e.g.

$$u(x) = \begin{bmatrix} u_r(x) \\ u_g(x) \\ u_b(x) \end{bmatrix}$$

transform according to defining repr. of $SU(3)$

$$\mathcal{L}_{QCD} = \sum_q \{ \bar{q}(x)\, i\gamma^\mu (\partial_\mu + i g_s \frac{\lambda_a}{2} G^a_\mu(x)) q(x) - m_q \bar{q}(x) q(x) \} + \mathcal{L}_G$$

$q = u, d, \ldots$

gluon field $G^a_\mu(x)$, $a = 1, \ldots, 8$

back to the general case:

local gauge transformation

$$\psi(x) \to U(x)\psi(x)$$

$$U(x) = e^{-i\alpha_a(x) T_a}$$

$$T_a A^a_\mu(x) =: A_\mu(x) \longrightarrow A'_\mu(x)$$

condition for invariance of $\mathcal{L}_\psi$ under local gauge transformation

$$D_\mu \psi = (\partial_\mu + ig A_\mu)\psi \to (\partial_\mu + ig A'_\mu) U \psi$$

$$= U \partial_\mu \psi + (\partial_\mu U)\psi + ig A'_\mu U \psi$$

$$= U \left[\partial_\mu \psi + ig\left(U^{-1} A'_\mu U - \frac{i}{g} U^{-1} \partial_\mu U\right)\right]\psi$$

$$= U D_\mu \psi$$

$\uparrow$

if $U^{-1} A'_\mu U - \frac{i}{g} U^{-1} \partial_\mu U = A_\mu$

$\Updownarrow$

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1}$$

A'_μ can be written in the form $A'_\mu = T_a A'^a_\mu$

proof:

we consider the expression $X(t) := e^{-itY} X e^{itY}$

X, Y are quadratic matrices

$X(t)$ fulfils the differential equation

$$\dot{X}(t) = -i [Y, X(t)]$$

with the initial condition $X(0) = X$

I define the linear operator

$$(\mathrm{ad}\,Y)\,X := [Y, X]$$

acting on the space of matrices;

with this definition, the differential equation can be written in the form

$$\dot{X}(t) = -i\,(\mathrm{ad}\,Y)\,X(t)\,,$$

the solution is given by

$$X(t) = e^{-it\,\mathrm{ad}\,Y} \underbrace{X(0)}_{X}$$

$$\Rightarrow\; e^{-iY} X\, e^{iY} = X(1) = e^{-i\,\mathrm{ad}\,Y} X$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\mathrm{ad}\,Y)^n\, X$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \underbrace{[Y, [Y, \ldots [Y, X]\ldots]}_{n\text{-fold commutator}}$$

we apply this formula to our problem:

$X = T_a, \quad Y = \alpha_\beta T_\beta =: \alpha \cdot T$

$$e^{-i\alpha.T}\, T\, e^{i\alpha.T} = e^{-i\,ad(\alpha.T)}\, T =$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [\alpha.T, [\alpha.T, \dots [\alpha.T, T_a]\dots]$$

$$(ad\,\alpha.T)\, T_a = [\alpha_\beta T_\beta, T_a] = \alpha_\beta\, i f_{\beta ac}\, T_c$$

$$= T_c\, \alpha_\beta \underbrace{(-i f_{\beta ca})}_{(t_\beta)_{ca}} = T_c\, \alpha_\beta (t_\beta)_{ca}$$

$$\Rightarrow e^{-i\alpha.T}\, T_a\, e^{i\alpha.T} = T_c \left(e^{-i\alpha_\beta t_\beta}\right)_{ca}$$

$$\Rightarrow U A_\mu U^{-1} = e^{-i\alpha.T}\, T_a A^a_\mu\, e^{i\alpha.T}$$

$$= T_c \left(e^{-i\alpha_\beta t_\beta}\right)_{ca} A^a_\mu$$

second term:

$$U(x + t e(\mu))\, U(x)^{-1} = e^{-i\alpha_a(x + t e(\mu))\, T_a}\, e^{i\alpha_\beta(x)\, T_\beta} =$$

↑ unit vector in μ-direction

$$= e^{-i\beta_a(x,t)T_a}$$

$$U(x)\,U(x)^{-1} = \mathbb{1} \Rightarrow \quad \beta_a(x,0) = 0$$

$$\frac{d}{dt}\, U(x + t e_{(\mu)})\, U(x)^{-1}\Big|_{t=0} = \left(\partial_\mu U(x)\right) U(x)^{-1}$$

$$= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\beta_a(x,t)\, T_a\right)^n \Big|_{t=0} =$$

$$= -i\, \frac{\partial \beta_a(x,t)}{\partial t}\Big|_{t=0} T_a$$

$$\Rightarrow \quad A'_\mu = U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U)\, U^{-1}$$

can be written as $A'_\mu = A'^{a}_{\mu}\, T_a$

- gauge boson sector

generalized field strength tensor

$$F^a_{\mu\nu}\, T_a =: F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]$$

$$= \partial_\mu A^a_\nu\, T_a - \partial_\nu A^a_\mu\, T_a + ig\, \underbrace{[A^b_\mu T_b,\, A^c_\nu T_c]}_{i f_{bca}\, T_a\, A^b_\mu\, A^c_\nu}$$

$$= \underbrace{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu \right)}_{F^a_{\mu\nu}} T_a$$

$F_{\mu\nu}$ transforms homogeneously: $F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$ (exercise)

gauge boson term in the Lagrangean:

$$\mathrm{Tr}(T_a T_b) = \Lambda \delta_{ab}$$

$$\to \mathcal{L}_A = -\frac{1}{4\Lambda} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} F^a_{\mu\nu} F_a{}^{\mu\nu}$$

→ kinetic terms + self couplings of the gauge bosons