11. Abelian gauge field

Quantum electrodynamics (QED) with a single charged Dirac field (e.g. e^-) described by

$$\mathcal{L}_{QED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - j^\mu A_\mu$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ field strength tensor of photon field A_μ

$j^\mu = q\bar{\psi}\gamma^\mu\psi$ electromagnetic current (density)

q = elm charge of Dirac field ψ

($q = -e$ for electron field)

remark: $\bar{\psi}(i\not{\partial} - m)\psi - j^\mu A_\mu =$

$$= \bar{\psi}\left[i\gamma^\mu \underbrace{(\partial_\mu + iqA_\mu)}_{\text{covariant derivative}} - m\right]\psi$$

$\mathcal{L}_{QED}$ is invariant under the local $U(1)$ gauge transformation $\psi(x) \to e^{-iq\Lambda(x)}\psi(x)$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$

$$\partial_\mu \psi(x) \to \partial_\mu \left(e^{-iq\Lambda(x)} \psi(x) \right) =$$

$$= e^{-iq\Lambda(x)} \left(\partial_\mu \psi(x) - iq\, \partial_\mu \Lambda(x)\ \psi(x) \right)$$

$$iq\, A_\mu(x)\, \psi(x) \to iq\, \left(A_\mu(x) + \partial_\mu \Lambda(x) \right) e^{-iq\Lambda(x)} \psi(x)$$

$$\Rightarrow D_\mu \psi := (\partial_\mu + iq\, A_\mu)\psi \to e^{-iq\Lambda(x)} D_\mu \psi$$

$$\Rightarrow \bar{\psi}\, i \not{D} \psi \to \bar{\psi}\, i \not{D} \psi \quad \text{invariant}$$

$\bar{\psi}\psi$, $F_{\mu\nu} F^{\mu\nu}$ —"—

$\Rightarrow \mathcal{L}_{QED}$ invariant

$U(1)$ = <u>abelian</u> gauge group

QED = <u>abelian gauge theory</u>

still to be determined: <u>photon propagator</u>

consider elm. field in the presence of a classical current density J^μ ($\underline{\partial_\mu J^\mu = 0}$)

→ generating functional

$$Z[J] = \frac{1}{N} \int [dA^\mu] \; e^{i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right)}$$

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right)$$

$$= \int d^4x \left[-\frac{1}{2} A^\mu \left(-g_{\mu\nu} (\Box - i\varepsilon) + \partial_\mu \partial_\nu \right) A^\nu - J_\mu A^\mu \right]$$

↑ partial integration

we could try again the usual trick: shift of variables $A^\mu = A'^\mu + B^\mu$ → B^μ should fulfil

$$\left(g_{\mu\nu} \Box - \partial_\mu \partial_\nu \right) B^\nu = J_\mu$$

propagator $D^{\nu\rho}(x)$ = inverse of the differential operator $g_{\mu\nu} \Box - \partial_\mu \partial_\nu$:

$$\left(g_{\mu\nu} \Box - \partial_\mu \partial_\nu \right) D^{\nu\rho}(x) = \underbrace{g_\mu{}^\rho}_{\delta_\mu{}^\rho} \delta^{(4)}(x)$$

Fourier representation $D^{\nu\sigma}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{D}^{\nu\sigma}(k)$

→ in momentum space, we get the equation

$$(-g_{\mu\nu} k^2 + k_\mu k_\nu)\, \tilde{D}^{\nu\sigma}(k) = g_\mu{}^\sigma$$

we discuss the more general problem:
find the inverse of

$$T_{\mu\nu} = a(k^2)\, k_\mu k_\nu + b(k^2)\, g_{\mu\nu}$$

ansatz for $(T^{-1})^{\nu\sigma}$:

$$(T^{-1})^{\nu\sigma} = A(k^2)\, k^\nu k^\sigma + B(k^2)\, g^{\nu\sigma}$$

$$\Rightarrow (a k_\mu k_\nu + b g_{\mu\nu})(A k^\nu k^\sigma + B g^{\nu\sigma}) = g_\mu{}^\sigma$$

$$a A k^2 k_\mu k^\sigma + a B k_\mu k^\sigma + b A k_\mu k^\sigma + b B g_\mu{}^\sigma = g_\mu{}^\sigma$$

$$\Rightarrow \quad B = \frac{1}{b} \;, \quad a A k^2 + a B + b A = 0$$

$$A\,(a k^2 + b) = -\frac{a}{b} \quad \Rightarrow \quad A = -\frac{a}{b}\,\frac{1}{a k^2 + b^2}$$

$$\Rightarrow \quad (T^{-1})^{\nu\sigma} = -\frac{a\,k^\nu k^\sigma}{b\,(a k^2 + b)} + \frac{g^{\nu\sigma}}{b}$$

$\Rightarrow$ T^{-1} exists if $a k^2 + b \neq 0$ and $b \neq 0$

in the case of the photon field we have:

$a = 1$, $b = -k^2$ $\Rightarrow$ $a k^2 + b = 0$

$\Rightarrow$ inverse of $(-g_{\mu\nu} k^2 + k_\mu k_\nu)$ does

<u>not</u> exist

indeed: $(-g_{\mu\nu} k^2 + k_\mu k_\nu)\, k^\nu = 0 \qquad (k^\nu \neq 0)$

can already be seen in the space-time representation:

$$(g_{\mu\nu}\,\Box - \partial_\mu \partial_\nu)\,\partial^\nu \Lambda = \Box\,\partial_\mu \Lambda - \partial_\mu \Box \Lambda = 0$$

way out: gauge fixing term in the Lagrangean

(Fermi's trick)

gauge parameter

$$\mathcal{L} \to \mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \underbrace{- \frac{\xi}{2} (\partial_\mu A^\mu)^2}_{\text{gauge fixing term}} - J_\mu A^\mu$$

the action can now be written in the form

$$S = \int d^4x \left\{ \frac{1}{2} A^\mu \left[g_{\mu\nu} \Box - (1-\xi) \partial_\mu \partial_\nu \right] A^\nu - J_\mu A^\mu \right\}$$

the equation $\left[g_{\mu\nu} \Box - (1-\xi) \partial_\mu \partial_\nu \right] B^\nu = J_\mu$

can now be inverted:

$$T_{\mu\nu} = -k^2 g_{\mu\nu} + (1-\xi) k_\mu k_\nu$$

$$a = 1-\xi \,, \quad b = -k^2$$

$$\Rightarrow \quad (T^{-1})^{\nu\rho} = -\frac{1}{k^2} \left[g^{\nu\rho} - \left(1 - \frac{1}{\xi}\right) \frac{k^\nu k^\rho}{k^2} \right]$$

Feynman gauge ($\xi = 1$): $(T^{-1})^{\nu\rho} = -\dfrac{g^{\nu\rho}}{k^2}$

$$D^{\nu\rho}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \underbrace{\frac{-1}{k^2+i\varepsilon}\left[g^{\nu\rho} - \left(1-\frac{1}{\xi}\right)\frac{k^\nu k^\rho}{k^2}\right]}_{\widetilde{D}^{\nu\rho}(k)}$$

$$S'_{eff} = \int d^4x \left\{ \frac{1}{2} A^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\partial_\mu \partial_\nu \right] A^\nu - J_\mu A^\mu \right\}$$

$A_\mu = A'_\mu + B_\mu$ ↓ =

$$\int d^4x \left\{ \frac{1}{2} A'^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\partial_\mu \partial_\nu \right] A'^\nu \right.$$

$$+ A'^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\partial_\mu \partial_\nu \right] B^\nu - J_\mu A'^\mu$$

$$\left. + \frac{1}{2} B^\mu \left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi)\partial_\mu \partial_\nu \right] B^\nu - J_\mu B^\mu \right\}$$

$$\left[g_{\mu\nu}(\Box - i\varepsilon) - (1-\xi) \right] B^\nu = J_\mu$$

i.e. $B^\mu(x) = \int d^4y\, D^{\mu\nu}(x-y)\, J_\nu(y)$

→ terms linear in A'^μ disappear

$$\Rightarrow S_{eff} = \int d^4x \left\{ \frac{1}{2} A'^\mu \left[g_{\mu\nu} (\Box - i\varepsilon) - (1-\xi) \partial_\mu \partial_\nu \right] A'^\nu \right.$$

$$\left. - \frac{1}{2} J_\mu B^\mu \right\}$$

$$= \int d^4x \; \frac{1}{2} A'^\mu \left[g_{\mu\nu} (\Box - i\varepsilon) - (1-\xi) \partial_\mu \partial_\nu \right] A'^\nu$$

$$- \frac{1}{2} \int d^4x \, d^4y \; D^{\mu\nu}(x-y) \, J_\mu(x) \, J_\nu(y)$$

$$\Rightarrow Z[J] = e^{-\frac{i}{2} \int d^4x \, d^4y \; D^{\mu\nu}(x-y) J_\mu(x) J_\nu(y)}$$

<u>remark:</u> $Z[J]$ is independent of the gauge parameter ξ for a <u>conserved</u> current ($\partial^\mu J_\mu = 0$): to see this, we compute

$$\int d^4y \; D^{\mu\nu}(x-y) \, J_\nu(y)$$

and convince ourselves that this expression is independent of ξ if $\partial^\mu J_\mu = 0$

$$J_\nu(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, \tilde{J}_\nu(p)$$

$$\partial^\nu J_\nu(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, (-ip^\nu)\, \tilde{J}_\nu(p)$$

$$\partial^\nu J_\nu = 0 \quad \Leftrightarrow \quad p^\nu \tilde{J}_\nu(p) = 0$$

$$\Rightarrow \int d^4y\; D^{\mu\nu}(x-y)\, J_\nu(y) =$$

$$= \int d^4y \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{-1}{k^2+i\varepsilon} \left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2} \right]$$

$$\times \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, \tilde{J}_\nu(p)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{-1}{k^2+i\varepsilon} \left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2} \right] \tilde{J}_\nu(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{-g^{\mu\nu}}{k^2+i\varepsilon}\, \tilde{J}_\nu(k)$$

$\uparrow$ $k^\nu \tilde{J}_\nu(k) = 0$

$$\Rightarrow \int d^4x\, d^4y\; D^{\mu\nu}(x-y)\, J_\mu(x)\, J_\nu(y)$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{-\tilde{J}^\mu(k)}{k^2+i\varepsilon} \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \tilde{J}_\mu(p)$$

$$= -\int \frac{d^4k}{(2\pi)^4}\, \frac{\tilde{J}^\mu(k)\, \tilde{J}_\mu(-k)}{k^2+i\varepsilon}$$

$J^\mu(x)$ real $\Rightarrow$ $\tilde{J}^\mu(-k) = \tilde{J}^\mu(k)^*$

$$\Rightarrow \int d^4x\, d^4y\; D^{\mu\nu}(x-y)\, J_\mu(x)\, J_\nu(y)$$

$$= -\int \frac{d^4k}{(2\pi)^4}\, \frac{\tilde{J}^\mu(k)\, \tilde{J}_\mu(k)^*}{k^2+i\varepsilon}$$

$$\Rightarrow Z[J] = \langle 0\,\text{out} \mid 0\,\text{in}\rangle_J = e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \frac{\tilde{J}^\mu(k)\, \tilde{J}_\mu(k)^*}{k^2+i\varepsilon}}$$

path integral quantization of an abelian gauge field

reminder : $S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

$$= \frac{1}{2} \int d^4x \, A^\mu \underbrace{\left(g_{\mu\nu} \Box - \partial_\mu \partial_\nu\right)}_{\text{not invertible}} A^\nu$$

→ usual "trick" (shift of integration variables) not applicable

reason : $S[A_\mu + \partial_\mu \Lambda] = S[A_\mu]$

(action invariant under abelian gauge transformations)

in the expression

$$\langle 0 | T \, O[A] | 0 \rangle = \frac{\int [dA] \, e^{iS[A]} \, O[A]}{\int [dA] \, e^{iS[A]}}$$

↑ gauge invariant observable

integration performed also over gauge-equivalent field configurations

goal: we want to factorize out the redundant part of the integration measure

→ Faddeev - Popov Quantization

possible procedure:

$$\Delta[A,\omega] \int [d\lambda]\, \delta(\partial_\mu A^\mu + \Box\lambda - \omega) = 1$$

this equation defines $\Delta[A,\omega]$

remark: $\Box \to \Box + i\varepsilon$ tacitly assumed to ensure uniqueness of the solution of

$$\partial^\mu A_\mu + (\Box + i\varepsilon)\lambda - \omega = 0$$

$$\to \quad \lambda = \frac{1}{\Box + i\varepsilon}\,(\omega - \partial_\mu A^\mu)$$

$$\Delta[A,\omega]^{-1} = \int [d\lambda]\, \delta(\partial . A + \Box\lambda - \omega)$$

$$= \Delta[0,\ \omega - \partial . A]^{-1}$$

transformation of the integration variable (shift): $\lambda = \lambda' + \mu$

↑ new variable of integration

$$\rightarrow \Delta[A,\omega]^{-1} = \int [d\lambda']\, \delta(\partial . A + \Box\lambda' + \Box\mu - \omega)$$

$$= \Delta[A, 0]^{-1}$$ (by an appropriate choice of μ)

$\Rightarrow$ $\Delta[A,\omega] = \Delta[0,0]$ is <u>independent</u> of A_μ, ω (this is <u>not</u> the case for <u>nonabelian</u> gauge theories)

this result can also be seen in the following way:

$$\int [d\varphi]\, \delta(\varphi) = 1$$

variable transformation $\varphi = \partial . A + (\Box + i\varepsilon)\lambda - \omega$

$\rightarrow$ unique mapping $\lambda = \frac{1}{\Box + i\varepsilon} (\varphi + \omega - \partial . A)$

$$\int [d\lambda] \left(\det \frac{\delta \varphi(x)}{\delta \lambda(y)} \right) \delta \left(\partial . A + (\Box + i\varepsilon)\lambda - \omega \right) = 1$$

$$\frac{\delta \varphi(x)}{\delta \lambda(y)} = (\Box + i\varepsilon)\, \delta(x - y)$$

$\Rightarrow \Delta [A, \omega] = \det (\Box + i\varepsilon)$ <u>independent of A, ω</u>

$$\rightarrow \quad \langle 0|\, T\, \mathcal{O}[A]\, |0\rangle$$

$$= \frac{\int [dA\, d\lambda]\ \Delta[0,0]\ \delta(\partial.A + \Box\lambda - \omega)\ e^{iS[A]}\ \mathcal{O}[A]}{\int [dA\, d\lambda]\ \underbrace{\Delta[0,0]\ \delta(\partial.A + \Box\lambda - \omega)}_{1}\ e^{iS[A]}}$$

$$= \frac{\int [dA\, d\lambda]\ \delta(\partial.A + \Box\lambda - \omega)\ e^{iS[A]}\ \mathcal{O}[A]}{\int [dA\, d\lambda]\ \delta(\partial.A + \Box\lambda - \omega)\ e^{iS[A]}}$$

variable transformation $A_\mu = A'_\mu + \partial_\mu \Lambda$

(gauge transformation)

$$S[A] = S[A'] \ , \quad \mathcal{O}[A] = \mathcal{O}[A']$$

(gauge invariant quantities)

→ λ disappear by an appropriate <u>choice of Λ</u>

$$\rightarrow \quad \langle 0|T\mathcal{O}[A]|0\rangle = \frac{\cancel{\int [d\lambda]} \int [dA]\ \delta(\partial.A - \omega)\ e^{iS[A]}\ \mathcal{O}[A]}{\cancel{\int [d\lambda]} \int [dA]\ \delta(\partial.A - \omega)\ e^{iS[A]}}$$

desired factorization ↑

numerator and denominator <u>independent</u> of ω → integrate these expressions over a <u>weight function</u>

$$\langle 0|T\,O[A]|0\rangle =$$

$$= \frac{\int [d\omega]\; e^{-\frac{i}{2\xi}\int d^4x\,\omega(x)^2} \int [dA]\,\delta(\partial.A-\omega)\, e^{iS[A]}\, O[A]}{\int [d\omega]\; e^{-\frac{i}{2\xi}\int d^4x\,\omega(x)^2} \int [dA]\,\delta(\partial.A-\omega)\, e^{iS[A]}}$$

$$= \frac{\int [dA]\; e^{i\left[S[A]-\int d^4x\,\frac{1}{2\xi}(\partial.A)^2\right]}\, O[A]}{\int [dA]\; e^{i\left[S[A]-\int d^4x\,\frac{1}{2\xi}(\partial.A)^2\right]}}$$

$$\rightarrow\; S_{eff}[A] = S[A] - \frac{1}{2\xi}\int d^4x\;(\partial.A)^2$$