6. Renormalizability

Our one-loop calculation showed: two-point and four-point function requires renormalization.

Six-point function (at one-loop) determined by convergent integral

\[ \sim \int \frac{d^d k}{k^6} \sim \Lambda^{d-6} \]

\[ d-6 = -2 \quad \text{for} \quad d = 4 \]

\[ \Lambda \quad \text{UV-cutoff} \]

Generally: $\Gamma_n$ finite for $n \geq 6$ at one-loop.

Higher orders (two-loop, ...) field, mass, and renormalization of a sufficient to obtain finite results for all observable quantities
\[ \phi = \sqrt{Z} \phi \, , \quad m^2 = m_{ph}^2 - 8m^2, \quad \lambda = Z \lambda_{ph} \]

\[ L = \frac{1}{2} \left( \partial_{\mu} \phi \partial^{\mu} \phi - m^2 \phi^2 \right) - \frac{\lambda_{ph}}{4!} \phi^4 \]

\[ = \frac{1}{2} \left( \partial_{\mu} \phi_{ph} \partial^{\mu} \phi_{ph} - m_{ph}^2 \phi_{ph}^2 \right) - \frac{\lambda_{ph}}{4!} \phi_{ph}^4 \]

\[ + \frac{1}{2} \left( Z - 1 \right) \partial_{\mu} \phi_{ph} \partial^{\mu} \phi_{ph} \]

\[ - \frac{1}{2} \left[ (Z - 1) m_{ph}^2 - Z 8m^2 \right] \phi_{ph}^2 \]

\[ - \left( Z^2 Z_{\phi} - 1 \right) \lambda_{ph} \frac{1}{4!} \phi_{ph}^4 \]

"physical" perturbation theory (using \( m_{ph}, \lambda_{ph} \))

with counterterms \( \frac{1}{2} A \partial_{\mu} \phi_{ph} \partial^{\mu} \phi_{ph} - \frac{1}{2} B \phi_{ph}^2 - \frac{C \phi_{ph}^4}{4!} \)

coefficients \( A, B, C \) determined iteratively

(order by order in perturbation theory)

by three renormalization conditions:
Fourier transform of two-point function has

1. a pole at $p^2 = m^2_{ph}$

2. with residue equal to one

3. scattering amplitude evaluated at $s = 0$, to
   should be equal to $\lambda_{ph}$

\text{example: one-loop calculation in "physical"}
perturbation theory

\text{two-loop function:}

$$\langle 0 | T \Phi_{ph}(x) \Phi_{ph}(0) | 0 \rangle =$$

$$= \int \frac{d^d k}{(2\pi)^d} \ e^{-ikx} \left\{ \frac{1}{i} \frac{1}{m^2_{ph} - k^2 - i\varepsilon} + \frac{\lambda_{ph} \Delta(0)}{2} \frac{1}{(m^2_{ph} - k^2 - i\varepsilon)^2} + \left( -i A R^2 + i B \right) \frac{1}{(m^2_{ph} - k^2 - i\varepsilon)^2} \right\}$$

$$\Rightarrow A = \mathcal{O} + \mathcal{O}(\lambda^2_{ph}) \quad B = \frac{i \lambda_{ph}}{2} \Delta(0) + \mathcal{O}(\lambda^2_{ph})$$
four-point function

\[ \rightarrow \text{scattering amplitude} \]

\[ M(s,t) = -\alpha_{ph} + \frac{\alpha_{ph}^2}{2} \left[ B(s, m_{ph}^2) + B(t, m_{ph}^2) + B(u, m_{ph}^2) \right] - C \]

\[ \Rightarrow C = \frac{\alpha_{ph}^2}{2} \left[ \text{Re} \, B(s, m_{ph}^2) + B(t, m_{ph}^2) + B(u, m_{ph}^2) \right] + O(\alpha_{ph}^3) \]

all observables made finite by this procedure
(in all orders of the perturbative expansion)

\[ \phi^4 \text{-theory is an example of a renormalizable QFT} \]
(determined by a finite number of parameters)

general proof of the renormalizability of \(\phi^4\)-theory is beyond the scope of this course

\[ \rightarrow \text{here we just analyze the general structure of the divergences occurring at higher orders} \]
\[ \text{remember: } (2\pi)^d \delta^{(d)}(R_1 + \ldots + R_n) \prod_n (R_1, \ldots, R_n) = \]
\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{-i\alpha}{4!} \right)^k \int \cdots \int d^d y_1 \ldots d^d y_k \]
\[ \left\langle \phi(R_1) \ldots \phi(R_n) \phi(y_1)^k \ldots \phi(y_k) \right\rangle \]

study one-particle irreducible diagram with \( n \) external momenta and \( R \) vertices

one-particle irreducible graph: does not fall apart if one of the internal lines is cut

draw examples:

\begin{align*}
\text{one-particle irreducible} & \quad \text{not one-particle irreducible} \\
\text{number of internal lines: } I &= \frac{1}{2} (4R - n) \\
\text{number of loops: } L &= I - (R - 1) \\
\Rightarrow L &= R - \frac{n}{2} + 1 \\
& \quad \text{overall energy-momentum conservation}
\end{align*}
degree of divergence

one-particle irreducible graph (euclidean):

\[ \sim \int d^{d_{1}} \ldots d^{d_{L}} \prod_{i=1}^{L} \frac{1}{m^{2} + q_{i}^{2}} \sim \Lambda^{Ld-2I} \text{ for } Ld+2I \]

\[ q_{i} = q_{i} (l_{1}, \ldots, l_{L}; R_{1}, \ldots, R_{n}) \]

external momenta

superficial degree of divergence:

\[ \omega = Ld - 2I = (R - \frac{n}{2} + 1) d - 4R + n \]

\[ = d + (d-4)R - \frac{1}{2} (d-2) n \]

\[ d = 4 : \quad \omega = 4 - n \]

\( \omega < 0 \) necessary condition for convergence of integral (but not sufficient)
counterexample:

\[ w = -2 \]

but diagram divergent because of divergent subgraph

\[ w_{\text{sub}} = 4 - 2 = 2 \quad \text{(quadr. div.)} \]

Heinberg's theorem: integral convergent if the graph as well as all of its subgraphs posses a negative superficial degree of divergence; divergences occur only if the graph or some of its subgraphs have \( w \geq 0 \)

in our case: \( w = 4 - n \)

\( n = 0 \) → vacuum bubbles do not contribute to perturbation series

\( n = 2, n = 4 \) only possibility for divergent graphs and subgraphs
we had seen: divergent one-loop contributions to two- and four-point functions had the same structure as the tree terms $\phi^2$ (mass term) and $\phi^4$ (interaction term) \rightarrow divergences could be absorbed by renormalization of bare parameters $m, \alpha$

generalization of this property holds true also at higher orders: divergences only associated with local terms $\partial^2 \phi, \phi^2, \phi^4$

remark: investigation of general case rather complicated, mainly due to overlapping divergences

example:

$\rightarrow \phi^4$ - theory is renormalizable