4. Perturbation theory

theory with interaction term:

\[ S[\varphi] = S_0[\varphi] + S_{\text{int}}[\varphi] \]

\[ \text{action of free theory} \]
\[ \text{bilinear in } \varphi \]

\[ \text{example: } \varphi^4 - \text{theory} \]

\[ S_{\text{int}}[\varphi] = -\frac{\lambda}{4!} \int dx \varphi(x)^4 \]

\[ \text{remark: anticipate already dimensional regularization (}dx^4 \rightarrow dx^4\) \]

\[ \text{perturbation series:} \]

\[ e^{iS} = e^{iS_0} e^{iS_{\text{int}}} = e^{iS_0} \sum_{k=0}^{\infty} \frac{ik^k}{k!} S_{\text{int}}^k \]

\[ \langle 0 | T \phi(x_1) \ldots \phi(x_n) | 0 \rangle = \]

\[ = \frac{\int [d\varphi] e^{iS} \phi(x_1) \ldots \phi(x_n)}{\int [d\varphi] e^{iS}} \]

\[ \text{vacuum of interacting theory} \]
\[
\left< e^{iS_{\text{int}}[\varphi]} \varphi(x_1) \ldots \varphi(x_n) \right> = \frac{\int [d\varphi] e^{iS_{\text{int}}[\varphi]} e^{iS_{\phi}[\varphi]} \varphi(x_1) \ldots \varphi(x_n)}{\int [d\varphi] e^{iS_{\phi}[\varphi]}}
\]

where \( \left< F[\varphi] \right> := \frac{\int [d\varphi] e^{iS_{\phi}[\varphi]} F[\varphi]}{\int [d\varphi] e^{iS_{\phi}[\varphi]}} \)

Gaussian mean value of the function \( F[\varphi] \)

perturbative expansion:

\[
\left< e^{iS_{\text{int}}[\varphi]} \varphi(x_1) \ldots \varphi(x_n) \right> = \sum_{k=0}^{\infty} \frac{i^k}{k!} \left< S_{\text{int}}[\varphi]^k \varphi(x_1) \ldots \varphi(x_n) \right>
\]
in the case of a $\phi^4$-theory:

$$
\left< e^{i \int \mathcal{L}} \phi(x_1) \ldots \phi(x_n) \right> = \\
= \sum_{k=0}^{\infty} \frac{(-i \lambda)^k}{k! (4!)^k} \int dy_1 \ldots dy_k \left< \phi(y_1)^4 \ldots \phi(y_k)^4 \phi(x_1) \ldots \phi(x_n) \right>
$$

we know already:

$$
\left< \phi(x_1) \ldots \phi(x_r) \right> = \begin{cases} 
0 & \text{for } n \text{ odd} \\
\sum_{\text{pairs}} \frac{1}{i} \Delta(x_i - x_j) \ldots \frac{1}{i} \Delta(x_{r-1} - x_r) & \text{for } n \text{ even}
\end{cases}
$$

difference to previously considered case:

in addition to the product $\phi(x_1) \ldots \phi(x_n)$ with different space-time points $x_1, \ldots, x_n$ also terms from the interaction $\sim \phi(y)^4$ with fields at the same space-time point $y$

pairing rule ( Wick's theorem) $\rightarrow$ form all possible pairings of the $n + 4k$ fields and replace the various pairs by $\frac{1}{i} \Delta(x_i - x_j)$
two types of points:

vertices \( y_1, \ldots, y_k \)

external points \( x_1, \ldots, x_n \)

two types of lines connecting the points:

internal lines: connect vertices

external lines: at least one endpoint is external point

examples:

\[
\begin{align*}
\text{external points:} & \quad x_1 \quad \quad x_2 \quad \quad x_3 \\
\text{external line:} & \quad y_1 \quad \quad y_2 \\
\text{internal line:} & \quad x_4 \quad \quad x_5 \quad \quad x_6 \\
\text{connected graph:} & \quad y_1 \quad \quad y_2 \\
\text{disconnected graph:} & \quad x_1 \quad \quad x_2 \quad \quad x_3 \quad \quad x_4 \quad \quad x_5 \quad \quad x_6
\end{align*}
\]
UV divergences $\rightarrow$ UV-regularization necessary

Bulk force method: stay in $d=4$, $\int d^4k$ made finite by cut-off $|k^\mu| < \Lambda \rightarrow$ violates Lorentz invariance (in intermediate steps of the calculation) $\rightarrow$ Lorentz invariance only recovered in final results

Dimensional regularization respects Lorentz invariance (and other symmetries of the theory under consideration)

Renormalization $\rightarrow$ finite result for observables in the limit $d \rightarrow 4$ (or $\Lambda \rightarrow \infty$)

IR divergences generated by $\int d^d y$ ... in (disconnected)

Vacuum bubbles

Example: $\Delta(y-y) = \Delta(0)$

$\Delta(x)$ singular for $x^2 = 0$

UV divergence for $d \rightarrow 4$

$\int d^d y \Delta(\mathbf{0}) \Delta(\mathbf{0})$

$\sim V T$ (volume of space-time)
contributions of disconnected vacuum bubbles cancel in the ratio

\[
\frac{\left\langle e^{iS_{\text{int}}} \phi(x_1) \ldots \phi(x_n) \right\rangle}{\left\langle e^{iS_{\text{int}}} \right\rangle}
\]

\[\rightarrow \text{IR divergences do not cause any problems}
\]
(situation a bit more complicated in massless theories)

\text{n-point functions in } \phi^4\text{-theory:}

non-vanishing result only for even } n \text{ (reason: symmetry } \phi \rightarrow -\phi \text{ of } \phi^4\text{-theory)}

two-point function to } O(\lambda) :

\[
\left\langle e^{iS_{\text{int}}} \phi(x_1) \phi(x_2) \right\rangle
\]

\[= \left\langle \phi(x_1) \phi(x_2) \right\rangle - \frac{i\lambda}{4!} \int d^4y \left\langle \phi(x_1) \phi(x_2) \phi(y)^4 \right\rangle + O(\lambda^2)
\]

\[x_1 \quad + \quad x_1 \quad + \quad x_2 \quad + \quad O(\lambda^2)
\]
\[ \left\langle \varphi(x_1) \varphi(x_2) \varphi(y)^4 \right\rangle = \]
\[ = 4 \frac{1}{i} \Delta(x_1-y) 3 \frac{1}{i} \Delta(x_2-y) \frac{1}{i} \Delta(o) \]
\[ \langle 0 | T \varphi(x_1) \varphi(x_2) | 0 \rangle = \frac{1}{i} \Delta(x_1-x_2) \]
\[ + \frac{\lambda}{2} \Delta(o) \int dy \int \frac{d^dR}{(2\pi)^d} \frac{1}{m^2 - R^2 - i\varepsilon} \]
\[ = \frac{1}{i} \int \frac{d^dR}{(2\pi)^d} \frac{e^{-i R(x_1-x_2)}}{m^2 - R^2 - i\varepsilon} \]
\[ + \frac{\lambda}{2} \Delta(o) \int dy \int \frac{d^dR_1}{(2\pi)^d} \frac{e^{-i R_1(x_1-y)}}{m^2 - R_1^2 - i\varepsilon} \int \frac{d^dR_2}{(2\pi)^d} \frac{e^{-i R_2(x_1-y)}}{m^2 - R_2^2 - i\varepsilon} \]
\[ = \frac{1}{i} \int \frac{d^dR}{(2\pi)^d} \frac{e^{-i R(x_1-x_2)}}{m^2 - R^2 - i\varepsilon} \]
\[ + \frac{\lambda}{2} \Delta(o) \int \frac{d^dR_1}{(2\pi)^d} \frac{d^dR_2}{(2\pi)^d} \frac{e^{-i R_1 x_1} e^{-i R_2 x_1}}{(m^2 - R_1^2 - i\varepsilon)(m^2 - R_2^2 - i\varepsilon)} \]
\[ + O(\lambda^2) \]
\[ = \frac{4}{i} \int \frac{d^dR}{(2\pi)^d} \frac{e^{-i R(x_1-x_2)}}{m^2 - R^2 - i\varepsilon} + \frac{\lambda}{2} \Delta(o) \int \frac{d^dR}{(2\pi)^d} \frac{e^{-i R(x_1-x_2)}}{(m^2 - R^2 - i\varepsilon)^2} \]
\[ + O(\lambda^2) \]
\begin{align*}
\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle &= \\
&= \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{\delta_{ij}}{m^2 - k^2 - i\epsilon} \left( \frac{1}{\sqrt{2}} \frac{1}{m^2 - k^2 - i\epsilon} \right)^2 \left( 1 + \frac{i\alpha_0}{2} \right) + \mathcal{O}(\alpha^2) \\
&= \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{\delta_{ij}}{m^2 - k^2 - i\epsilon} \left( \frac{1}{\sqrt{2}} \frac{1}{m^2 - k^2 - i\epsilon} \right)^2 + \mathcal{O}(\alpha^2) \\
&= \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{\delta_{ij}}{m^2 - i\alpha_0} - \frac{k^2}{2} - i\epsilon + \mathcal{O}(\alpha^2) \\
\text{general structure of two-point function (arbitrary order in perturbative expansion):} \\
\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle &= \\
&= \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{\delta_{ij}}{m^2 + \Sigma(k^2) - k^2 - i\epsilon} \\
\text{remark: one-loop contribution to } \Sigma(k^2) \text{ in } \phi^4 \text{-theory} \\
\text{independent of } k^2; \kbar^2 \text{-dependence arises only at two loops in the case of } \phi^4 \text{-theory}
pole of the propagator (in momentum space) shifted by higher order corrections

physical interpretation:

\[
\langle 0 | T \Phi(x) \Phi(0) | 0 \rangle = \\
= \Theta(x^0) \langle 0 | \Phi(x) \Phi(0) | 0 \rangle + \Theta(-x^0) \langle 0 | \Phi(0) \Phi(x) | 0 \rangle \\
= \Theta(x^0) \sum_\alpha \langle 0 | \Phi(x) | \alpha \rangle \langle \alpha | \Phi(0) | 0 \rangle + \Theta(-x^0) \sum_\alpha \langle 0 | \Phi(0) | \alpha \rangle \langle \alpha | \Phi(x) | 0 \rangle \\
= \Theta(x^0) \sum_\alpha \langle 0 | e^{iP_x} \Phi(0) e^{-iP_x} | \alpha \rangle \langle \alpha | \Phi(0) | 0 \rangle + \Theta(-x^0) \sum_\alpha \langle 0 | \Phi(0) | \alpha \rangle \langle \alpha | e^{iP_x} \Phi(0) e^{-iP_x} | 0 \rangle \\
= \Theta(x^0) \sum_\alpha e^{-iP_x x} |\langle 0 | \Phi(0) | \alpha \rangle|^2 + \Theta(-x^0) \sum_\alpha e^{iP_x x} |\langle 0 | \Phi(0) | \alpha \rangle|^2
\]
\[ \sum_{n=1}^{\infty} \alpha | \alpha \rangle \langle \alpha | = \langle 0 | + \int d\mu(p) \langle 1p | \langle p | \]

\[ + \sum_{n=2}^{\infty} \frac{1}{n!} \int d\mu(p_1) \ldots d\mu(p_n) \langle 1p_1 \ldots p_n | \langle \text{in} | \langle \text{out} | p_1 \ldots p_n | \text{in} \rangle \]

In our case no contribution from vacuum state as \[ \langle 0 | \phi(0) | 0 \rangle = 0 \quad (\text{in theories with} \]

\[ \langle 0 | \phi(0) | 0 \rangle \neq 0 \quad \text{take} \quad \phi(x) - \langle 0 | \phi(0) | 0 \rangle \quad \text{instead of} \quad \phi(x) \]

\[ \Rightarrow \langle 0 | T \phi(x) \phi(0) | 0 \rangle = \]

\[ = \Theta(x^0) \int d\mu(p) e^{-ip.x} | \langle 0 | \phi(0) | p \rangle |^2 \]

\[ + \Theta(-x^0) \int d\mu(p) e^{ip.x} | \langle 0 | \phi(0) | p \rangle |^2 \]

\[ + \text{contributions from intermediate states} \]

\[ \text{with} \quad n \geq 2 \]
\( \langle 0 \mid \phi(0) \mid p \rangle \) is independent of \( p \) because of Lorentz invariance:

\( \mid p \rangle = \sqrt{m_p^2 + p^2}, \hat{p} \rangle \) can be obtained by a Lorentz transformation acting on \( \mid m_p, \sigma \rangle \):

\( \mid p \rangle = \sqrt{m_p^2 + p^2}, \hat{p} \rangle = U(L) \mid m_p, \sigma \rangle \),

where \( L(m_p, \sigma) = (\sqrt{m_p^2 + p^2}, \hat{p}) \)

\[ \Rightarrow \langle 0 \mid \phi(0) \mid p \rangle = \langle 0 \mid \phi(0) U(L) \mid m_p, \sigma \rangle \]

\[ = \langle 0 \mid U(L)^{-1} \phi(0) U(L) \mid m_p, \sigma \rangle = \]

\[ U(L) \mid 0 \rangle = \mid 0 \rangle \]

\[ = \langle 0 \mid \phi(0) \mid m_p, \sigma \rangle \quad \text{independent of } p \]

\( \phi \) is a scalar field

\( U(L)^{-1} \phi(x) U(L) = \phi(L^{-1}x) \)

\( Z := |\langle 0 \mid \phi(0) \mid p \rangle|^2 \)

with a suitable definition of the phase of \( \mid p \rangle \):

\( |Z| = \langle 0 \mid \phi(0) \mid p \rangle \)
\[
\langle 0 | T \phi(x) \phi(0) | 0 \rangle =
\]

\[
= \Xi \left\{ \Theta(x^0) \int d\mu(p) e^{-ip \cdot x} + \Theta(-x^0) \int d\mu(p) e^{ip \cdot x} \right\} + \ldots
\]

\[
= \frac{1}{i} \Delta(x; m^2_{ph})
\]

\[
= \Xi \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik \cdot x}}{m^2_{ph} - k^2 - i\varepsilon} + \ldots
\]

pole of the two-point function in momentum space determines the physical mass \( m_{ph} \)

\[
\Rightarrow m^2 + \Sigma(m^2_{ph}) - m^2_{ph} = 0
\]

residue of the pole at \( k^2 = m^2_{ph} \) occurring in

\[
\frac{1}{M^2 + \Sigma(k^2) - k^2 - i\varepsilon}
\]

may be worked out by expanding the function \( \Sigma(k^2) \) around this point:

\[
\Sigma(k^2) = \Sigma(m^2_{ph}) + (k^2 - m^2_{ph}) \Sigma'(m^2_{ph}) + O[(k^2 - m^2_{ph})^2] \Rightarrow \Xi = \frac{1}{1 - \Sigma'(m^2_{ph})}
\]