3. Functional integral (path integral)

quantum mechanics:

$$\langle 0 | T\, Q(t_1) \dots Q(t_n) | 0 \rangle = \frac{1}{\mathcal{N}} \int [dq]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1) \dots q(t_n)$$

operators ground state c-numbers

$$S[q] = \int_{-\infty}^{+\infty} dt \left[\frac{1}{2} \dot{q}(t)^2 - V(q(t)) \right] \quad \text{classical action}$$

$$\mathcal{N} = \int [dq]\, e^{\frac{i}{\hbar} S[q]} \qquad \text{normalization constant}$$

$[dq]$ integration measure (classical theory $\rightarrow$ quantum theory)

metric in the space of all paths $\rightarrow$ volume element

consider two neighbouring paths $q(t)$, $q(t) + dq(t)$

$(\text{distance})^2$ between two paths:

$$ds^2 = \int dt\, [dq(t)]^2$$

$$q(t) = \sum_{i=1}^{\infty} q_i\, u_i(t) \qquad \{u_i\}_{i=1}^{\infty} \text{ complete set of ortho-normal functions } \int dt\, u_i(t) u_j(t) = \delta_{ij}$$

<u>regularization</u>: restrict the space of paths to those for which only the first N components $q_1, \dots, q_N$ are different from zero:

$$q(t) = q_1 u_1(t) + \dots + q_n u_n(t)$$

$$ds^2 = (dq_1)^2 + \dots + (dq_N)^2$$

$$\Rightarrow \quad [dq]_{reg} = dq_1 \dots dq_N$$

$$\int [dq] \dots = \lim_{N\to\infty} \int dq_1 \dots dq_N \dots$$

important property of the measure: <u>translation invariance</u>

$q(t) = q'(t) + k(t)$ (arbitrary function $k(t)$)

→ metric unchanged ⇒ volume element induced by metric unchanged ⇒ $[dq] = [dq']$

explicit construction of measure in terms of a metric not indispensable → translation invariance sufficient relation between classical mechanics and quantum theory:

rapid oscillation of $e^{\frac{i}{\hbar}S[q]}$ when $q \to q + \delta q$ in the classical limit $\hbar \to 0$, except in the neighbourhood of the path where the phase is stationary:

$$\left.\frac{\delta S[q]}{\delta q(t)}\right|_{q=q_{cl}} = 0$$

$$\Updownarrow$$

$$m\ddot{q}_{cl}(t) = -V'(q_{cl}) \quad \text{classical equation of motion}$$

→ significant contribution to the path integral only from this region around $q_{cl}(t)$

→ laws of classical mechanics can be formulated as action principle

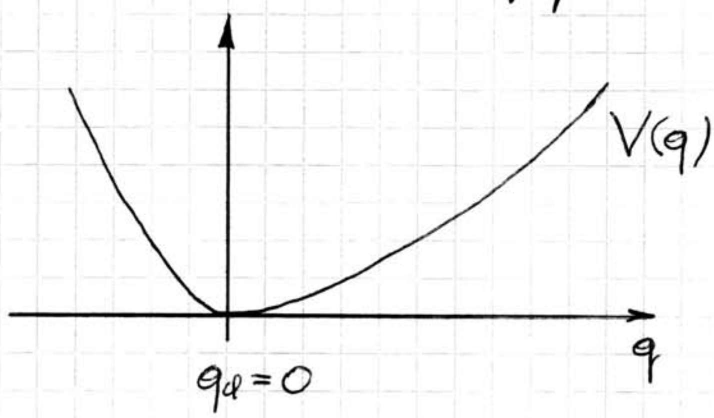
region which dominates the path integral for $\hbar \to 0$ characterized by $|S[q] - S[q_{cl}]| \lesssim \hbar$: only there the contributions interfere constructively

$q(t) = q_{cl}(t) + \delta q_{cl}(t)$

$S[q] - S[q_{cl}] \sim (\delta q)^2$ (no linear term)

→ path integral dominated by those paths for which $q(t) = q_{cl}(t) + \mathcal{O}(\sqrt{\hbar})$

relevant classical solution for vacuum expectation values: classical groundstate characterized by $V'(q_{cl}) = 0$; choose coordinates such that minimum of potential occurs at $q_{cl} = 0$:



$\Rightarrow$ $q(t) \sim \sqrt{\hbar}$ generate significant contributions to the path integral

$$\Rightarrow \langle 0 | T\, Q(t_1)\, Q(t_2) | 0 \rangle = \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\, q(t_2)$$

$$\sim \mathcal{O}(\hbar)$$

exercises: calculate $\langle 0 | T\, Q(t_1)\, Q(t_2) | 0 \rangle$ for the harmonic oscillator using both the operator approach and the path integral method

quantum field theory:

$$\langle 0 | T\, \phi(x_1) \ldots \phi(x_n) | 0 \rangle = \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{iS[\varphi]}\, \varphi(x_1) \ldots \varphi(x_n)$$

operators — c-number fields

remark: back to $\hbar = 1$

$$\mathcal{N} = \int [d\varphi]\, e^{iS[\varphi]}$$

$S[\varphi]$ ... classical action of the model, i.e.

$$S[\varphi] = \int d^4x\, \frac{1}{2} \left[\partial_\mu \varphi(x) \partial^\mu \varphi(x) - m^2 \varphi(x)^2 \right]$$

(free scalar field)

integration measure $[d\varphi]$: immediate generalization of the one in quantum mechanics

distance between two neighbouring field configurations $\varphi(x)$, $\varphi(x) + d\varphi(x)$ given by

$$ds^2 = \int d^4x\, [d\varphi(x)]^2$$

$\rightarrow$ $[d\varphi]$ is volume element induced by this metric shift $\varphi(x) = \varphi'(x) + k(x)$ leaves distance unchanged

$$\Rightarrow \quad [d\varphi] = [d\varphi']$$

generating functional

$$Z[f] = \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{iS[\varphi]}\, e^{i\int d^4x\, f(x)\varphi(x)}$$

$f(x)$: external field (source)

$$= \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{iS[\varphi]} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1\, f(x_1)\varphi(x_1) \ldots \int d^4x_n\, f(x_n)\varphi(x_n)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n\, f(x_1) \ldots f(x_n) \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{iS[\varphi]} \varphi(x_1) \ldots \varphi(x_n)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n\, f(x_1) \ldots f(x_n) \langle 0 | T \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$

$$=: \langle 0 | T\, e^{i\int d^4x\, f(x)\phi(x)} | 0 \rangle$$

n-point functions can be obtained as functional derivatives of $Z[f]$ with respect to the external field, taken at $f=0$:

$$\langle 0 | T \phi(x_1) \ldots \phi(x_n) | 0 \rangle = \frac{1}{i^n} \left. \frac{\delta^n Z[f]}{\delta f(x_1) \ldots \delta f(x_n)} \right|_{f=0}$$

remark: $\dfrac{\delta f(x)}{\delta f(y)} = \delta^{(4)}(x-y)$; $Z[0] = 1$

computation of $Z[f]$ for the free scalar field:

$$Z[f] = \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{i\int d^4x \left[\underbrace{\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x)}_{-\frac{1}{2}\varphi(x)\,\Box\,\varphi(x)} - \frac{m^2}{2}\varphi(x)^2 + f(x)\varphi(x)\right]}$$

(partial integration)

Gaussian functional integral with purely imaginary exponent (Fresnel type integral) $\rightarrow$ introduce damping factor by $m^2 \rightarrow m^2 - i\varepsilon$ ($\varepsilon > 0$); $\varepsilon \rightarrow 0$ after integration

shift of integration variable: $\varphi(x) = \varphi'(x) + k(x)$

new integration variable

choose $k(x)$ such that terms linear in $\varphi'(x)$ disappear:

$$\int d^4x \left[-\frac{1}{2}\varphi(x)\,(\Box + m^2)\,\varphi(x) + f(x)\varphi(x)\right] =$$

$$= \int d^4x \left[-\frac{1}{2}(\varphi'(x) + k(x))\,(\Box + m^2)\,(\varphi'(x) + k(x)) + f(x)\,(\varphi'(x) + k(x))\right] =$$

$$= \int d^4x \, \Big[\underbrace{-\frac{1}{2}\varphi'(x)\,(\Box + m^2 - i\varepsilon)\,\varphi'(x)}_{\text{quadratic in } \varphi'(x)}$$

$$\underbrace{-\frac{1}{2}\varphi'(x)\,(\Box + m^2 - i\varepsilon)\,k(x) - \frac{1}{2}k(x)\,(\Box + m^2 - i\varepsilon)\,\varphi'(x) + f(x)\,\varphi'(x)}_{\text{linear in } \varphi'(x)}$$

$$\underbrace{-\frac{1}{2}k(x)\,(\Box + m^2 - i\varepsilon)\,k(x) + f(x)\,k(x)}_{\text{independent of } \varphi'(x)} \Big]$$

$\Rightarrow$ $k(x)$ *must fulfil* $\quad (\Box + m^2 - i\varepsilon)\,k(x) = f(x)$

$$\Rightarrow \quad Z[f] = \frac{1}{N} \int [d\varphi']\, e^{-\frac{i}{2}\int d^4x\, \varphi'(x)\,(\Box + m^2 - i\varepsilon)\,\varphi'(x)}$$

$$\times\, e^{\frac{i}{2}\int d^4x\, f(x)\, k(x)}$$

but $\quad N = \int [d\varphi]\, e^{-\frac{i}{2}\int d^4x\, \varphi(x)\,(\Box + m^2 - i\varepsilon)\,\varphi(x)}$

$$\Rightarrow \quad Z[f] = e^{\frac{i}{2}\int d^4x\, f(x)\, k(x)}$$

$\rightarrow$ *have to find solution of* $\;(\Box + m^2 - i\varepsilon)\,k(x) = f(x)$

the corresponding homogeneous equation does not admit physically meaningful solutions :

$(p^0)^2 - \vec{p}^{\,2} = m^2 - i\varepsilon$ implies that for real three-momentum, the frequency contains a small imaginary part

$\rightarrow$ $e^{\pm ipx}$ explodes either for $x^0 \rightarrow +\infty$ or $x^0 \rightarrow -\infty$

$\Rightarrow$ $k(x)$ is uniquely determined by $(\Box + m^2 - i\varepsilon) k(x) = f(x)$

we know already: $(\Box + m^2 - i\varepsilon)\,\Delta(x) = \delta^{(4)}(x)$

$$\Rightarrow \quad k(x) = \int d^4y\, \Delta(x-y)\, f(y)$$

$$\Rightarrow \quad Z[f] = \langle 0|\, T\, e^{i\int d^4x\, f(x)\phi(x)}\, |0\rangle$$

$$= e^{\frac{i}{2}\int d^4x\, d^4y\, f(x) f(y) \Delta(x-y)}$$

expansion of the r.h.s. in powers of f contains only even terms $\Rightarrow$ $\langle 0|\, T\phi(x_1)\ldots\phi(x_n)\,|0\rangle = 0$ if n is odd (e.g. $\langle 0|\phi(x)|0\rangle = 0$)

two-point function: $\frac{i^2}{2!}\, \langle 0|\, T\, \phi(x)\, \phi(y)\, |0\rangle = \frac{i}{2}\, \Delta(x-y)$

$$\Rightarrow \quad \langle 0|\, T\phi(x)\phi(y)\, |0\rangle = \frac{1}{i}\, \Delta(x-y)$$

for n even:

$$\int d^4x_1 \dots d^4x_n \, f(x_1) \dots f(x_n) \, \frac{i^n}{n!} \, \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

$$= \int d^4x_1 \dots d^4x_n \, f(x_1) \dots f(x_n) \, \frac{i^p}{2^p p!} \, \Delta(x_1 - x_2) \dots \Delta(x_{n-1} - x_n)$$

$$p = \frac{n}{2}$$

integrand on the l.h.s. <u>symmetric</u> with respect to an interchange of the arguments, but integrand on the r.h.s. is <u>not</u>

only <u>totally symmetric parts</u> of the two integrands are the same!

$\Delta(-x) = \Delta(x) \Rightarrow$ it suffices to consider all possible partitions of the set $1, \dots, n$ into pairs $(i_1, i_2), \dots, (i_{n-1}, i_n) \rightarrow (n-1)(n-3) \dots 1 = (n-1)!!$ such pairings $\rightarrow$ use $2^p p! \, (2p-1)!! = (2p)!$

$$\frac{i^{2p}}{(2p)!} \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{i^p}{2^p p!} \, \frac{2^p p!}{(2p)!} \sum_{\text{pairings}} \Delta(x_{i_1} - x_{i_2}) \dots$$
$$\dots \Delta(x_{i_{n-1}} - x_{i_n})$$

$$\Rightarrow \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle = \sum_{\text{pairings}} \frac{1}{i}\Delta(x_{i_1}-x_{i_2})\ldots\frac{1}{i}\Delta(x_{i_{n-1}}-x_{i_n})$$

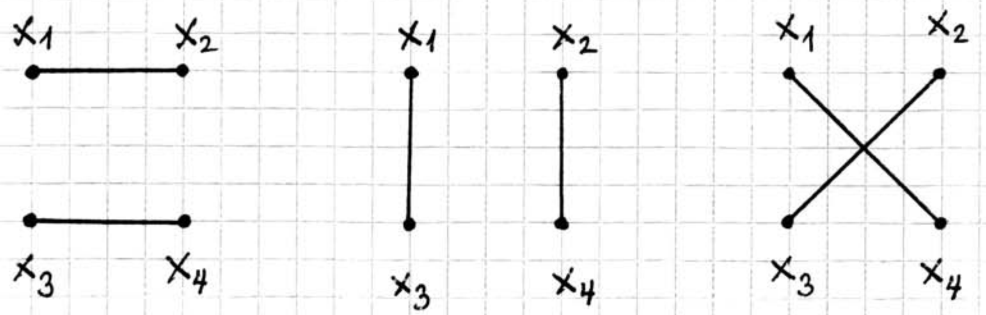
graphic representation :

(i) represent the arguments $(x_1, \ldots, x_n)$ of the n-point function by points in the plane (non-vanishing Green function only for even n)

(ii) draw the graphs that arise by decomposing the set of points into pairs indicating this with a line between the two points

(iii) for each line connecting the points x, y denote a factor $\frac{1}{i}\Delta(x-y)$; the contribution associated with a given graph is the product of the factors coming from the various lines

(iv) final result is the sum of the contributions from the different graphs example: four-point function of a free scalar field



$$\langle 0 | T\, \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4)\, | 0 \rangle =$$

$$= \frac{1}{i} \Delta(x_1 - x_2)\; \frac{1}{i} \Delta(x_3 - x_4)$$

$$+ \frac{1}{i} \Delta(x_1 - x_3)\; \frac{1}{i} \Delta(x_2 - x_4)$$

$$+ \frac{1}{i} \Delta(x_1 - x_4)\; \frac{1}{i} \Delta(x_2 - x_3)$$

$(4-1)!! = 3$ different pairings $\rightarrow$ 3 graphs