## 2. Scalar fields (spin 0)

real scalar field $\phi(x) = \phi(x)^*$

$$\phi'(x') = \phi(x) \qquad x' = Lx + a \qquad (L \in \mathcal{L}_+^\uparrow)$$

- free scalar field $\rightarrow$ field equation $(\Box + m^2)\phi(x) = 0$

(Klein-Gordon equation)

action integral $S = \int d^4x \, \underbrace{\frac{1}{2}\left(\partial^\mu\phi\,\partial_\mu\phi - m^2\phi^2\right)}_{\mathcal{L}}$

Lagrange density

equation of motion $\partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} = \frac{\partial\mathcal{L}}{\partial\phi}$

$$\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} = \partial^\mu\phi \;,\; \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \;\Rightarrow\; (\Box + m^2)\phi = 0$$

Lagrangian $\rightarrow$ Hamiltonian

$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$ canonical momentum conjugate to $\phi$

→ Hamilton density $\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}[\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2]$

no <u>explicit</u> x - dependence of $\mathcal{L}$ → energy-momentum conservation
(inv. under space-time translations)

energy momentum tensor $\quad T_{\mu\nu} = \underbrace{\frac{\partial\mathcal{L}}{\partial\phi^{,\mu}}}_{\phi_{,\mu}} \phi_{,\nu} - g_{\mu\nu}\mathcal{L}$

$$\partial^\mu T_{\mu\nu} = 0$$

$\Rightarrow \quad P^\mu = \int d^3x \; T^{0\mu}(x) = \text{const.} \quad$ 4-momentum

$\vec{P} = -\int d^3x \; \pi \vec{\nabla}\phi \quad$ 3-momentum of scalar field

invariance under rotations → angular momentum cons.

$$\vec{L} = -\int d^3x \; \pi \; \vec{x} \times \vec{\nabla}\phi$$

<u>quantization</u> (canonical quantization)

equal time commutation relations

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\phi(t,\vec{x}), \phi(t,\vec{y})] = [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0$$

Fourier decomposition

$$\phi(x) = \underbrace{\int d\mu(p)}_{\frac{d^3p}{(2\pi)^3 2p^0}} \left[ a(p)\, e^{-ipx} + a(p)^\dagger e^{+ipx} \right]$$

$$p^0 = \sqrt{m^2 + \vec{p}^{\,2}} =: \omega(\vec{p})$$

$$p \cdot x = p^0 t - \vec{p} \cdot \vec{x}$$

$$a(p) = i \int d^3x\, e^{ipx} \overleftrightarrow{\partial_0} \phi(x) \qquad \text{(exercise)}$$

$$A \overleftrightarrow{\partial} B := A\, \partial B - (\partial A) B$$

$$\Rightarrow \quad [a(p), a(p')^\dagger] = \underbrace{(2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p} - \vec{p}\,')}_{\delta(p, p')}$$

$$[a(p), a(p')] = [a(p)^\dagger, a(p')^\dagger] = 0$$

$$H' = \int d^3x\, \mathcal{H} = \frac{1}{2} \int d\mu(p)\, p^0 \left\{ a(p)^\dagger a(p) + a(p) a(p)^\dagger \right\}$$

$$= \int d\mu(p)\, p^0\, a(p)^\dagger a(p) + \underbrace{\frac{1}{2} \int d\mu(p)\, p^0\, \delta(p, p)}_{E_{vac}}$$

vacuum energy $E_{vac} = \frac{1}{2} \int d^3p \; p^0 \; \delta^{(3)}(\vec{0})$

$$\delta^{(3)}(\vec{0}) = \lim_{\vec{p}' \to \vec{p}} \delta^{(3)}(\vec{p} - \vec{p}') = \lim_{\vec{p}' \to \vec{p}} \int d^3x \, \frac{e^{i(\vec{p}-\vec{p}')\vec{x}}}{(2\pi)^3}$$

$\rightarrow \frac{V}{(2\pi)^3}$ in finite volume $V$ (<u>IR divergence</u>)

$\rightarrow$ energy <u>density</u> $\varepsilon_{vac} = E_{vac}/V = \frac{1}{2(2\pi)^3} \int d^3p \; p^0$

<u>but</u> even energy density <u>UV divergent</u>!

$$\varepsilon_{vac} = \frac{1}{2(2\pi)^3} \underbrace{\int}_{\substack{|\vec{p}| \le \Lambda \\ \uparrow \\ \text{UV cut-off}}} d^3p \; \sqrt{\vec{p}^{\,2} + m^2} = \frac{4\pi}{2(2\pi)^3} \int_0^\Lambda dp \, p^2 \sqrt{p^2 + m^2}$$

$$\simeq \frac{1}{(2\pi)^2} \frac{\Lambda^4}{4} \xrightarrow[\Lambda \to \infty]{} \infty$$

vacuum energy $E_{vac}$ can be removed by "renormalization"

$$H' \rightarrow H = \int d\mu(p) \; p^0 \; a(p)^\dagger a(p)$$

can be formally achieved by <u>normal ordering</u> $:\mathcal{H}:$ of the energy density: rearrange the order of the factors such that all creation operators stand to the left of all annihilation operators

– <u>field momentum</u>: $\vec{P} = -\int d^3x\, \pi\, \vec{\nabla}\phi = \int d\mu(p)\, \vec{p}\, a(p)^\dagger a(p)$

$$+ \vec{P}_{vac}$$

where $\vec{P}_{vac} = \frac{1}{2} \underbrace{\int d^3p\ \vec{p}}_{\substack{0 \text{ for rotation invariant} \\ \text{regularization}}}\ \delta^{(3)}(\vec{0})$

$\vec{P}_{vac}$ automatically removed by normal ordering:

$$\vec{P} = -\int d^3x :\pi\, \vec{\nabla}\phi: = \int d\mu(p)\, a(p)^\dagger a(p)\, \vec{p}$$

$\rightarrow \quad P^\mu = \int d\mu(p)\, a(p)^\dagger a(p)\, p^\mu \qquad$ 4-momentum

$$\Rightarrow \quad [P^\mu, a(p)] = -p^\mu a(p)$$

$$[P^\mu, a(p)^\dagger] = p^\mu a(p)^\dagger$$

(exercises)

→ exponentiated form

$$e^{iPa}\,\phi(x)\,e^{-iPa} = \phi(x+a)$$

$P^\mu$ generates space-time translations

ground state (vacuum state) $|0\rangle$ characterized by

$$a(p)\,|0\rangle = 0 \qquad \forall\, p$$

$$\Rightarrow \qquad P^\mu\,|0\rangle = 0$$

one-particle momentum eigenstates : $|p\rangle = a(p)^\dagger\,|0\rangle$

$$P^\mu\,|p\rangle = \underbrace{P^\mu a(p)^\dagger}_{a(p)^\dagger P^\mu + p^\mu a(p)^\dagger}\,|0\rangle = p^\mu\,|p\rangle$$

normalization $\langle p'|p\rangle = \underbrace{(2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p}\,'-\vec{p})}_{\delta(p',p)}$

$$N = \int \underbrace{d\mu(p)\; a(p)^\dagger a(p)}_{dn(p)} \qquad \text{particle number operator}$$

n-particle energy-momentum eigenstates:

$$|p_1, p_2, \dots, p_n\rangle = a(p_1)^\dagger a(p_2)^\dagger \dots a(p_n)^\dagger |0\rangle$$

$$P^\mu |p_1, \dots, p_n\rangle = (p_1^\mu + \dots + p_n^\mu) |p_1, \dots, p_n\rangle$$

$$\langle p_1, \dots, p_n | k_1, \dots, k_n \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \delta(p_i, k_{\sigma(i)})$$

permutations of n elements

projection operator on subspace of n-particle states

$$P^{(n)} = \frac{1}{n!} \int d\mu(p_1) \dots d\mu(p_n) \, |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n|$$

normalized n-particle state:

$$|\psi^{(n)}\rangle = \frac{1}{n!} \int d\mu(p_1) \dots d\mu(p_n) \, |p_1, \dots, p_n\rangle \underbrace{\langle p_1, \dots, p_n | \psi^{(n)}\rangle}_{\psi^{(n)}(p_1, \dots, p_n)}$$

totally symmetric wave fctn (Bose statistics)

$$\langle \psi^{(n)} | \psi^{(n)} \rangle = 1 \Leftrightarrow \frac{1}{n!} \int d\mu(p_1) \dots d\mu(p_n) \, | \psi^{(n)}(p_1, \dots, p_n) |^2 = 1$$

Fock space

$$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \dots$$

spanned by $\quad |0\rangle \qquad |p\rangle \qquad |p_1, p_2\rangle \quad \dots$

$$\mathbb{1} = \sum_{n=0}^{\infty} P^{(n)} \quad , \quad P^{(0)} = |0\rangle\langle 0|$$

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n-point functions (Green functions, correlation functions)

vacuum expectation values of time-ordered products of field operators

$$\langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

$$T \phi(x_1) \dots \phi(x_n) = \phi(x_{i_1}) \dots \phi(x_{i_n})$$

$i_1, \dots, i_n$ permutation of $1, \dots, n$ such that

$$x^0_{i_1} > x^0_{i_2} > \dots > x^0_{i_n}$$

interacting theory → S-matrix elements can be extracted from n-point functions (LSZ)

free theory → n-point functions can be written as sum of products of two-point functions

→ consider $\langle 0 | T \phi(x) \phi(y) | 0 \rangle$

for free scalar field

→ plays central rôle in perturbation expansion of interacting theories

theory translation invariant → 2-point function depends only on difference $x-y$:

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \langle 0 | T \phi(x-y) \phi(0) | 0 \rangle$$

(free) propagator

$$\Delta(x) := i \langle 0 | T \phi(x) \phi(0) | 0 \rangle$$

remark: $\Delta(-x) = \Delta(x)$

$$\langle 0|T\phi(x)\,\phi(0)|0\rangle = \Theta(x^0)\,\langle 0|\phi(x)\,\phi(0)|0\rangle$$

$$+\ \Theta(-x^0)\,\langle 0|\phi(0)\,\phi(x)\,|0\rangle$$

$$\langle 0|\phi(x)\,\phi(0)\,|0\rangle =$$

$$= \langle 0|\int d\mu(p)\,[e^{-ipx}a(p) + e^{+ipx}\cancel{a(p)^\dagger}]$$

$$\int d\mu(k)\,[\cancel{a(k)} + a(k)^\dagger]\,|0\rangle$$

$$= \int d\mu(p)\,d\mu(k)\,e^{-ipx}\,\langle 0|\underbrace{a(p)\,a(k)^\dagger}_{[a(p),\,a(k)^\dagger] = \delta(p,k)}|0\rangle$$

$$= \int d\mu(p)\,e^{-ipx}$$

analogously: $\langle 0|T\phi(0)\phi(x)|0\rangle = \int d\mu(p)\,e^{ipx}$

$$\Rightarrow \quad \Delta(x) = i\,\Theta(x^0)\int d\mu(p)\, e^{-ipx} + i\,\Theta(-x^0)\int d\mu(p)\, e^{+ipx}$$

only "positive" frequences for $x^0 > 0$ $\qquad e^{-i\omega(\vec{p})t}$

— " — "negative" — " — — " — $x^0 < 0$ $\qquad e^{+i\omega(\vec{p})t}$

$$(\Box + m^2)\,\Delta(x) = \delta^{(4)}(x)$$

$\Delta(x)$ is <u>Green function</u> of Klein-Gordon equation with the following boundary conditions: only pos. frequ. for $x^0 > 0$ and only neg. frequ. for $x^0 < 0$

(<u>Feynman boundary conditions</u>)

$\Delta(x)$ can be written as Fourier integral

$$\Delta(x) = \int \frac{d^4p}{(2\pi)^4}\, \frac{e^{-ipx}}{m^2 - p^2 - i\varepsilon}$$

Feynman boundary conditions taken into account by $m^2 \to m^2 - i\varepsilon$ ; in <u>this</u> formula, $p^0$ is an <u>integration variable</u> and not $\sqrt{\vec{p}^{\,2} + m^2}$ !

complex scalar field

$$\phi(x) = \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)] \quad , \quad \phi_i^* = \phi_i$$

$$(\Box + m^2)\phi(x) = 0 \quad \Leftrightarrow \quad (\Box + m^2)\phi_i(x) = 0$$

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} (\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i^2) = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

conserved current $\quad j^\mu = i\,\phi^* \overleftrightarrow{\partial^\mu} \phi$

reason: $\mathcal{L}$ invariant under (global) $U(1)$ gauge transformation $\quad \phi(x) \to e^{i\alpha}\phi(x) \quad , \quad \alpha \in \mathbb{R}$

(exercises)

canonical quantization:

$$[a_i(p), a_j(p')^\dagger] = \delta_{ij}\, \delta(p, p')$$

(all other commutators vanish)

$$\phi_i(x) = \int d\mu(p)\, [a_i(p) e^{-ipx} + a_i(p)^\dagger e^{+ipx}$$

$$\Rightarrow \quad \phi(x) = \int d\mu(p) \; [a_+(p)\, e^{-ipx} + a_-(p)^\dagger e^{+ipx}]$$

$$a_+(p) = \frac{1}{\sqrt{2}} \; [a_1(p) + i\, a_2(p)]$$

$$a_-(p)^\dagger = \frac{1}{\sqrt{2}} \; [a_1(p)^\dagger + i\, a_2(p)^\dagger]$$

$$[a_+(p),\, a_+(p')^\dagger] = [a_-(p),\, a_-(p')^\dagger] = \delta(p,p')$$

(all other commutators vanish)

$$\Rightarrow \quad [\phi(x),\, \dot{\phi}^\dagger(y)]\Big|_{x^0=y^0} = i\, \delta^{(3)}(\vec{x}-\vec{y})$$

consistent with $\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \pi(x) = \dot{\phi}^\dagger(x)$

$$i\, \langle 0 | T\, \phi(x)\, \phi(y)^\dagger | 0 \rangle = \Delta(x-y) \quad \text{(exercise)}$$

particle number operator

$$N = \sum_{i=1}^{2} \int \underbrace{d\mu(p)\; a_i(p)^\dagger a_i(p)}_{dn_i(p)}$$

$$= \int \underbrace{d\mu(p)\; a_+(p)^\dagger a_+(p)}_{dn_+(p)} + \int \underbrace{d\mu(p)\; a_-(p)^\dagger a_-(p)}_{dn_-(p)}$$

charge operator

$$Q = \int d^3x \; :j^0(x): \; = \int d\mu(p)\, [a_+(p)^\dagger a_+(p) - a_-(p)^\dagger a_-(p)]$$

$$= \int [dn_+(p) - dn_-(p)] \qquad \text{(exercise)}$$

energy-momentum operator

$$P^\mu = \sum_{i=1}^{2} \int d\mu(p)\, a_i(p)^\dagger a_i(p)\, p^\mu$$

$$= \int d\mu(p)\, [a_+(p)^\dagger a_+(p) + a_-(p)^\dagger a_-(p)]\, p^\mu$$

$$\left.\begin{array}{l} [P^\mu, Q] = 0 \\ \\ [P^\mu, P^\nu] = 0 \end{array}\right\} \Rightarrow \exists \text{ ONB of eigenvectors of } P^\mu, Q$$

$$[Q, a_\pm(p)^\dagger] = \pm a_\pm(p)^\dagger$$

vacuum $|0\rangle$: $\qquad a_\pm(p)\,|0\rangle \quad \forall p$

$$\Rightarrow Q\,|0\rangle = 0$$

$$Q \; a_\pm(p)^\dagger |0\rangle = \pm \; a_\pm(p)^\dagger |0\rangle$$

$\Rightarrow \quad |p,\pm\rangle := a_\pm(p)^\dagger |0\rangle$ eigenstates of $Q$
with eigenvalues $\pm 1$

$a_\pm(p)^\dagger$ creates state with charge $\pm 1$
$a_\pm(p)$ destroys – " – – " – – " –

nonhermitean scalar field describes particle and the associated <u>antiparticle</u>

examples : $\pi^\pm$ ($Q$ = electromagnetic charge in units of elementary charge $e$)

$K^0 \overline{K^0}$ ("charge" = strangeness $S = \pm 1$)

<u>charge conjugation</u>
interchange particle $\leftrightarrow$ antiparticle

field operator

$$\phi(x) = \int d\mu(p) \, [a_+(p) e^{-ipx} + a_-(p)^\dagger e^{+ipx}]$$

charge conjugate field:

$$\phi^c(x) = \int d\mu(p) \, [a_-(p) e^{-ipx} + a_+(p)^\dagger e^{+ipx}]$$

$\mathcal{L}$ invariant under $\phi \to \phi^*$ (<u>discrete</u> symmetry)

$\exists$ unitary operator $\mathcal{C}$ $(\mathcal{C}^\dagger \mathcal{C} = \mathcal{C}\mathcal{C}^\dagger = \mathbb{1})$

with $\mathcal{C}\phi(x)\mathcal{C}^{-1} = \phi(x)^\dagger \Rightarrow \mathcal{C} j^\mu \mathcal{C}^{-1} = -j^\mu$

$$\Downarrow$$

$$\mathcal{C} Q \mathcal{C}^{-1} = -Q$$

$$\Updownarrow$$

$$\mathcal{C} a_\pm(p) \mathcal{C}^{-1} = a_\mp(p) \Leftrightarrow \mathcal{C} a_\pm(p)^\dagger \mathcal{C}^{-1} = a_\mp(p)^\dagger$$

$\Rightarrow \mathcal{C}|0\rangle = |0\rangle$ (phase can be absorbed in $|0\rangle$)

$$\mathcal{C}|p,\pm\rangle = \mathcal{C} a_\pm(p)^\dagger |0\rangle = a_\mp(p) \mathcal{C}|0\rangle =$$

$$= a_\mp |0\rangle = |p,\mp\rangle$$