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Quantization of the (free) electromagnetic field

Maxwell's equations (Heaviside system with $c = 1$)

$$\text{curl } \vec{E} = -\dot{\vec{B}} \quad \text{div } \vec{E} = \rho$$

$$\text{curl } \vec{B} = \vec{j} + \dot{\vec{E}} \quad \text{div } \vec{B} = 0$$

energy density $\mathcal{H} = \frac{1}{2} (\vec{E}^2 + \vec{B}^2)$

momentum density $\vec{S} = \vec{E} \times \vec{B}$

angular momentum density $\vec{x} \times \vec{S}$

$\text{div } \vec{B} = 0 \Rightarrow \vec{B} = \text{curl } \vec{A}$ vector potential $\vec{A}$

$\text{curl } (\vec{E} + \dot{\vec{A}}) = 0 \Rightarrow \vec{E} = -\text{grad } \phi - \dot{\vec{A}}$

scalar potential ϕ

observable (physical) fields $\vec{E}, \vec{B}$ unchanged
under gauge transformation

$$\phi \rightarrow \phi + \Lambda$$

$$\vec{A} \rightarrow \vec{A} - \text{grad } \Lambda$$

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gauge invariance allows choice of gauge condition
examples:

Lorentz gauge $\dot{\phi} + \text{div } \vec{A} = 0$

Coulomb gauge $\text{div } \vec{A} = 0$

free electromagnetic field ($\rho = 0, \vec{j} = 0$)

we choose the gauge condition $\phi = 0, \text{div } \vec{A} = 0$
remaining field equations:

$$\text{div } \vec{E} = \text{div } \dot{\vec{A}} = \frac{\partial}{\partial t} \underbrace{\text{div } \vec{A}}_0 = 0 \quad \checkmark$$

$$\text{curl } \vec{B} = \dot{\vec{E}} \Rightarrow \underbrace{\text{curl curl } \vec{A}}_{-\Delta + \text{grad div}} = -\ddot{\vec{A}}$$

$$\Rightarrow \square \vec{A} = 0 \quad \text{d'Alembert operator} \quad \square = \frac{\partial^2}{\partial t^2} - \Delta$$

mathematical problem: general solution of

$$\text{div } \vec{A} = 0, \quad \square \vec{A} = 0$$

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space coordinates $\vec{x}$ confined to finite box Ω
(volume $V = L_1 L_2 L_3$)

periodic boundary conditions for vector potential $\vec{A}$

complete system of orthonormal functions

$$\frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}}, \quad k_i = \frac{2\pi n_i}{L_i} \quad (n_i \in \mathbb{Z})$$

with respect to scalar product

$$(f|g) = \int_{\Omega} d^3x \, f(\vec{x})^* g(\vec{x})$$

Fourier series

$$f(\vec{x}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}} c_{\vec{k}}$$

$$c_{\vec{k}} = \int_{\Omega} d^3x \, \frac{e^{-i\vec{k}\vec{x}}}{\sqrt{V}} f(\vec{x})$$

⇒ Fourier decomposition of the vector potential

$$\vec{A}(t, \vec{x}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}} \vec{c}_{\vec{k}}(t)$$

$$\vec{A} \text{ real} \Rightarrow \vec{c}_{-\vec{k}} = \vec{c}_{\vec{k}}^*$$

$$\operatorname{div} \vec{A} = 0 \Rightarrow \vec{k} \cdot \vec{c}_{\vec{k}} = 0$$

$$\square \vec{A} = 0 \Rightarrow \ddot{\vec{c}}_{\vec{k}}(t) + \vec{k}^2 \vec{c}_{\vec{k}}(t) = 0$$

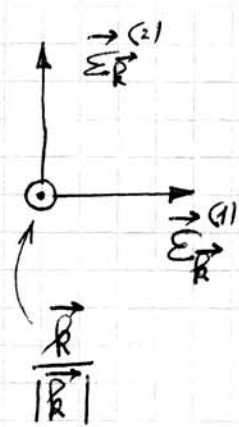
$$\Rightarrow \vec{c}_{\vec{k}}(t) = e^{-i\omega_{\vec{k}}t} \vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t} \vec{b}_{-\vec{k}}^*$$

$$\omega_{\vec{k}} := |\vec{k}|, \quad \vec{b}_{\vec{k}} \cdot \vec{k} = 0$$

$$\Rightarrow \vec{A}(t, \vec{x}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}} \left(e^{-i\omega_{\vec{k}}t} \vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t} \vec{b}_{-\vec{k}}^* \right)$$

$$= \sum_{\vec{k}} \frac{1}{\sqrt{V}} \left(\underbrace{e^{i\vec{k}\vec{x}} e^{-i\omega_{\vec{k}}t} \vec{b}_{\vec{k}}}_{\vec{b}_{\vec{k}}(t)} + \underbrace{e^{-i\vec{k}\vec{x}} e^{+i\omega_{\vec{k}}t} \vec{b}_{\vec{k}}^*}_{\vec{b}_{\vec{k}}(t)^*} \right)$$

condition $\vec{k} \cdot \vec{b}_k = 0$:



right-handed orthonormal basis of $\mathbb{R}^3$:
 $\underbrace{\vec{\epsilon}_k^{(1)}, \vec{\epsilon}_k^{(2)}, \frac{\vec{k}}{|\vec{k}|}}_{\text{real vectors}}$

$$|\vec{\epsilon}_k^{(\lambda)}| = 1$$

$$\vec{b}_k = b_k^{(1)} \vec{\epsilon}_k^{(1)} + b_k^{(2)} \vec{\epsilon}_k^{(2)}$$

real $\vec{\epsilon}_k^{(\lambda)}$ $\rightarrow$ decomposition in linearly polarized plane waves

circularly polarized plane waves :

$$\vec{\epsilon}_k^{(\lambda)} = -\lambda \frac{\vec{\epsilon}_k^{(1)} + i\lambda \vec{\epsilon}_k^{(2)}}{\sqrt{2}} \quad (\lambda = \pm 1)$$

$$\Rightarrow \vec{\epsilon}_k^{(\lambda)*} \cdot \vec{\epsilon}_k^{(\lambda')} = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}_k^{(\lambda)} \cdot \vec{k} = 0$$

$$\Rightarrow \vec{A}(t, \vec{x}) = \sum_{\vec{k}, \lambda} \frac{1}{V} \left(e^{i\vec{k}\vec{x}} \underbrace{e^{-i\omega_k t} b_k^{(\lambda)} \vec{\epsilon}_k^{(\lambda)}}_{b_k^{(\lambda)}(t)} + e^{-i\vec{k}\vec{x}} e^{i\omega_k t} b_k^{(\lambda)*} \vec{\epsilon}_k^{(\lambda)*} \right)$$

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exercises:energy of the electromagnetic field

$$H = \frac{1}{2} \int_{\Omega} d^3x (\vec{E}^2 + \vec{B}^2) = \sum_{\vec{k}, \lambda} \underbrace{2 \vec{k}^2 |b_{\vec{k}}^{(\lambda)}|^2}_{H_{\vec{k}}^{(\lambda)}}$$

momentum of the electromagnetic field

$$\vec{P} = \int_{\Omega} d^3x \vec{E} \times \vec{B} = 2 \sum_{\vec{k}, \lambda} \frac{\vec{k}}{|\vec{k}|} \vec{k}^2 |b_{\vec{k}}^{(\lambda)}|^2 = \sum_{\vec{k}, \lambda} \frac{\vec{k}}{|\vec{k}|} H_{\vec{k}}^{(\lambda)}$$

define real variables

$$q_{\vec{k}}^{(\lambda)}(t) = b_{\vec{k}}^{(\lambda)}(t) + b_{\vec{k}}^{(\lambda)*}(t),$$

$$p_{\vec{k}}^{(\lambda)}(t) = -i \omega_{\vec{k}} (b_{\vec{k}}^{(\lambda)}(t) - b_{\vec{k}}^{(\lambda)*}(t))$$

$$\Rightarrow \text{equation of motion } \dot{p}_{\vec{k}}^{(\lambda)}(t) = \dot{q}_{\vec{k}}^{(\lambda)}(t) = -\omega_{\vec{k}}^2 q_{\vec{k}}^{(\lambda)}(t)$$

harmonic oscillators

express $b_{\vec{k}}^{(\lambda)}(t)$ in terms of $q_{\vec{k}}^{(\lambda)}(t)$ and $p_{\vec{k}}^{(\lambda)}(t)$:

$$b_{\vec{k}}^{(\lambda)}(t) = \frac{1}{2} \left(q_{\vec{k}}^{(\lambda)}(t) + \frac{i}{\omega_{\vec{k}}} p_{\vec{k}}^{(\lambda)}(t) \right)$$

$\Rightarrow$ Hamiltonian

$$H = \frac{1}{2} \sum_{\vec{k}, \lambda} \left(p_{\vec{k}}^{(\lambda)2} + \omega_{\vec{k}}^2 q_{\vec{k}}^{(\lambda)2} \right)$$

Hamiltonian equations of motion

$$\dot{q}_{\vec{k}}^{(\lambda)} = \frac{\partial H}{\partial p_{\vec{k}}^{(\lambda)}} = p_{\vec{k}}^{(\lambda)}$$

$$\dot{p}_{\vec{k}}^{(\lambda)} = - \frac{\partial H}{\partial q_{\vec{k}}^{(\lambda)}} = - \omega_{\vec{k}}^2 q_{\vec{k}}^{(\lambda)}$$

free electromagnetic field $\equiv$ infinitely many
decoupled harmonic oscillators labeled
by $(\vec{k}, \lambda)$

quantization

$$[q_{\vec{k}}^{(\lambda)}, p_{\vec{k}'}^{(\lambda')}] = i \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'} \quad (\hbar = 1)$$

$$[q_{\vec{k}}^{(\lambda)}, q_{\vec{k}'}^{(\lambda')}] = [p_{\vec{k}}^{(\lambda)}, p_{\vec{k}'}^{(\lambda')}] = 0$$

annihilation operators

$$a_{\vec{k}}^{(\lambda)} := \frac{1}{\sqrt{2\omega_{\vec{k}}}} (\omega_{\vec{k}} q_{\vec{k}}^{(\lambda)} + i p_{\vec{k}}^{(\lambda)})$$

creation operators

$$a_{\vec{k}}^{(\lambda)\dagger} = \frac{1}{\sqrt{2\omega_{\vec{k}}}} (\omega_{\vec{k}} q_{\vec{k}}^{(\lambda)} - i p_{\vec{k}}^{(\lambda)})$$

$\Rightarrow$ commutation relations

$$[a_{\vec{k}}^{(\lambda)}, a_{\vec{k}'}^{(\lambda')\dagger}] = \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'}$$

$$[a_{\vec{k}}^{(\lambda)}, a_{\vec{k}'}^{(\lambda')}] = [a_{\vec{k}}^{(\lambda)\dagger}, a_{\vec{k}'}^{(\lambda')\dagger}] = 0$$

Hamilton operator

$$H' = \sum_{\vec{k}, \lambda} \omega_{\vec{k}} \left(a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)} + \frac{1}{2} \right)$$

$$[H', a_{\vec{k}}^{(\lambda)\dagger}] = \omega_{\vec{k}} a_{\vec{k}}^{(\lambda)\dagger}$$

groundstate $|0\rangle$ characterized by

$$a_{\vec{k}}^{(\lambda)} |0\rangle = 0 \quad \forall \vec{k}, \lambda \quad (\text{vacuum state})$$

$$H' |0\rangle = \underbrace{\frac{1}{2} \sum_{\vec{k}, \lambda} \omega_{\vec{k}}}_{\text{ground state energy (vacuum energy)}} |0\rangle$$

subtraction of vacuum energy:

$$H' \rightarrow H = \sum_{\vec{k}, \lambda} \omega_{\vec{k}} a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)}$$

$$H |0\rangle = 0$$

occupation number operator $N_{\vec{k}}^{(\lambda)} := a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)}$

measures the number of quanta (= photons)

with wave number vector $\vec{k}$ and polarization λ

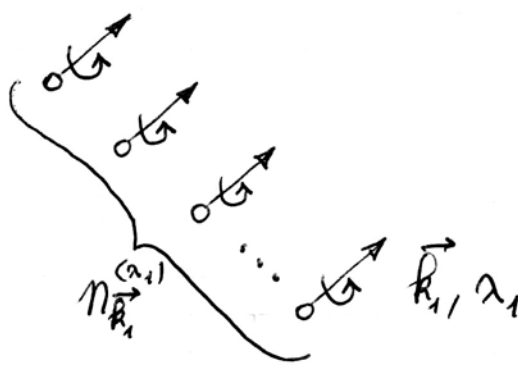
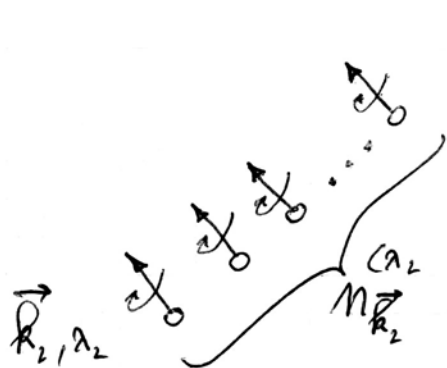
state vector

$$\frac{1}{\sqrt{n_{\vec{k}}^{(\lambda)}!}} (a_{\vec{k}}^{(\lambda)\dagger})^{n_{\vec{k}}^{(\lambda)}} |0\rangle =: |n_{\vec{k}}^{(\lambda)}\rangle$$

describes states with $n_{\vec{k}}^{(\lambda)}$ photons with quantum numbers $\vec{k}, \lambda$

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$$\frac{(a_{\vec{k}_1}^{(\lambda_1)})^\dagger)^{n_{\vec{k}_1}^{(\lambda_1)}}}{\sqrt{n_{\vec{k}_1}^{(\lambda_1)}!}} \frac{(a_{\vec{k}_2}^{(\lambda_2)})^\dagger)^{n_{\vec{k}_2}^{(\lambda_2)}}}{\sqrt{n_{\vec{k}_2}^{(\lambda_2)}!}} \dots |0\rangle = |n_{\vec{k}_1}^{(\lambda_1)} n_{\vec{k}_2}^{(\lambda_2)} \dots\rangle$$



$$H |n_{\vec{k}_1}^{(\lambda_1)}, n_{\vec{k}_2}^{(\lambda_2)}, \dots\rangle = (n_{\vec{k}_1}^{(\lambda_1)} \omega_{\vec{k}_1} + n_{\vec{k}_2}^{(\lambda_2)} \omega_{\vec{k}_2} + \dots) |n_{\vec{k}_1}^{(\lambda_1)}, n_{\vec{k}_2}^{(\lambda_2)}, \dots\rangle$$

total number of particles measured by

$$N = \sum_{\vec{k}, \lambda} N_{\vec{k}}^{(\lambda)} = \sum_{\vec{k}, \lambda} a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)}$$

$$N |n_{\vec{k}_1}^{(\lambda_1)}, n_{\vec{k}_2}^{(\lambda_2)}, \dots\rangle = \underbrace{(n_{\vec{k}_1}^{(\lambda_1)} + n_{\vec{k}_2}^{(\lambda_2)} + \dots)}_{\text{total number of photons in state } |n_{\vec{k}_1}^{(\lambda_1)}, n_{\vec{k}_2}^{(\lambda_2)}, \dots\rangle} |n_{\vec{k}_1}^{(\lambda_1)}, n_{\vec{k}_2}^{(\lambda_2)}, \dots\rangle$$

total number of photons
in state $|n_{\vec{k}_1}^{(\lambda_1)}, n_{\vec{k}_2}^{(\lambda_2)}, \dots\rangle$

momentum operator

$$\vec{P} = \sum_{\vec{k}, \lambda} \vec{k} a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)} = \sum_{\vec{k}, \lambda} \vec{k} N_{\vec{k}}^{(\lambda)}$$

remark: no problem with operator ordering in this case, as $\sum_{\vec{k}} \vec{k} = 0$

$$\vec{P} |\vec{k}, \lambda\rangle = \vec{k} |\vec{k}, \lambda\rangle$$

$\nwarrow$ photon momentum
one-photon state

$$|\vec{k}, \lambda\rangle := a_{\vec{k}}^{(\lambda)\dagger} |0\rangle$$

$$H |\vec{k}, \lambda\rangle = \omega_{\vec{k}} |\vec{k}, \lambda\rangle$$

$$\omega_{\vec{k}} = |\vec{k}| = \text{photon energy}$$

energy-momentum relation of massless particle

$$k^\mu = (\omega_{\vec{k}}, \vec{k}) \quad \text{photon } \text{four-momentum}$$

$$k \cdot x = k^\mu x_\mu = \omega_{\vec{k}} t - \vec{k} \cdot \vec{x} \quad \text{scalar product}$$

vector potential (field operator) expressed through creation and annihilation operators:

$$\vec{A}(t, \vec{x}) = \sum_{\vec{k}, \lambda} \frac{1}{\sqrt{2\omega_{\vec{k}} V}} \left(e^{-i\vec{k} \cdot \vec{x}} a_{\vec{k}}^{(\lambda)} \vec{\epsilon}_{\vec{k}}^{(\lambda)} + h.c. \right)$$

remark: $\vec{A}$ and $\dot{\vec{A}}$ fulfill equal-time commutation relation

$$[A_m(t, \vec{x}), \dot{A}_n(t, \vec{y})] = i \delta^{(3)}(\vec{x} - \vec{y}) \left(\delta_{mn} - \frac{k_m k_n}{k^2} \right)$$

limit $\Omega \rightarrow \mathbb{R}^3$ (continuum limit)

$$k_i = \frac{2\pi n_i}{L_i} \quad i = 1, 2, 3$$

$$n_i \in \mathbb{Z}$$

$$d^3 k = \frac{(2\pi)^3}{V} d^3 n \quad \Rightarrow \quad \sum_{\vec{k}} \rightarrow \int d^3 k \frac{V}{(2\pi)^3}$$

$$[a_{\vec{k}}^{(\lambda)}, a_{\vec{k}'}^{(\lambda')\dagger}] = \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'} = \delta_{\lambda\lambda'} \frac{1}{V} \int_{\Omega} d^3 x e^{i(\vec{k} - \vec{k}') \cdot \vec{x}}$$

$$\xrightarrow{\Omega \rightarrow \mathbb{R}^3} \delta_{\lambda\lambda'} \frac{1}{V} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$

$$a_{\vec{k}}^{(\lambda)} \xrightarrow{\Omega \rightarrow \mathbb{R}^3} \frac{1}{\sqrt{2\omega(\vec{k})V}} a(\vec{k}, \lambda)$$

$$\Rightarrow [a(\vec{k}, \lambda), a(\vec{k}', \lambda')^\dagger] = \underbrace{(2\pi)^3 2\omega(\vec{k}) \delta^{(3)}(\vec{k}-\vec{k}') \delta_{\lambda\lambda'}}_{=:\delta(\vec{k}, \vec{k}')} = \delta(\vec{k}, \vec{k}')$$

$$\vec{A}(t, \vec{x}) \rightarrow \sum_{\lambda} \int d^3k \frac{V}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{k})V}} \frac{1}{\sqrt{2\omega(\vec{k})V}} \times$$

$$\times (e^{-i\vec{k}\cdot\vec{x}} a(\vec{k}, \lambda) \vec{\epsilon}(\vec{k}, \lambda) + e^{i\vec{k}\cdot\vec{x}} a(\vec{k}, \lambda) \vec{\epsilon}(\vec{k}, \lambda)^*)$$

$$= \sum_{\lambda} \int \underbrace{\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}}_{d\mu(\vec{k})} (e^{-i\vec{k}\cdot\vec{x}} a(\vec{k}, \lambda) \vec{\epsilon}(\vec{k}, \lambda) + \text{h.c.})$$

$$H = \sum_{\vec{k}, \lambda} \omega_{\vec{k}} a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)} \rightarrow \sum_{\lambda} \int d^3k \frac{V}{(2\pi)^3} \omega(\vec{k}) \frac{1}{2\omega(\vec{k})V}$$

$$\times a(\vec{k}, \lambda)^\dagger a(\vec{k}, \lambda)$$

$$= \sum_{\lambda} \int \underbrace{\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}}_{d\mu(\vec{k})} \omega(\vec{k}) a(\vec{k}, \lambda)^\dagger a(\vec{k}, \lambda)$$

$$N = \sum_{\vec{k}, \lambda} a_{\vec{k}}^{(\lambda)\dagger} a_{\vec{k}}^{(\lambda)} \rightarrow \sum_{\lambda} \int d^3k \frac{V}{(2\pi)^3} \frac{1}{2\omega(\vec{k})V} a(\vec{k}, \lambda)^\dagger a(\vec{k}, \lambda)$$

$$= \sum_{\lambda} \int d\mu(\vec{k}) a(\vec{k}, \lambda)^{\dagger} a(\vec{k}, \lambda)$$

$$\vec{P} = \sum_{\vec{k}, \lambda} \vec{k} a_{\vec{k}}^{\dagger} a_{\vec{k}} \longrightarrow \sum_{\lambda} \int d^3k \frac{V}{(2\pi)^3} \vec{k} \frac{1}{2\omega(\vec{k})V} \\ \times a(\vec{k}, \lambda)^{\dagger} a(\vec{k}, \lambda)$$

$$= \sum_{\lambda} \int d\mu(\vec{k}) \vec{k} a(\vec{k}, \lambda)^{\dagger} a(\vec{k}, \lambda)$$

one-photon state with momentum $\vec{k}$ and polarization λ :

$$|\vec{k}, \lambda\rangle = a(\vec{k}, \lambda)^{\dagger} |0\rangle$$

$$H |\vec{k}, \lambda\rangle = \omega(\vec{k}) |\vec{k}, \lambda\rangle$$

$$\vec{P} |\vec{k}, \lambda\rangle = \vec{k} |\vec{k}, \lambda\rangle$$

covariant normalization:

$$\langle \vec{k}, \lambda | \vec{k}', \lambda' \rangle = \langle 0 | a(\vec{k}, \lambda) a(\vec{k}', \lambda')^{\dagger} | 0 \rangle \\ = \langle 0 | [a(\vec{k}, \lambda), a(\vec{k}', \lambda')^{\dagger}] | 0 \rangle \\ = (2\pi)^3 2\omega(\vec{k}) \delta^{(3)}(\vec{k} - \vec{k}') \delta_{\lambda\lambda'} \underbrace{\langle 0 | 0 \rangle}_1$$

normalized states $\rightarrow$ superpositions of momentum eigenstates

general one-photon state:

$$|f^{(1)}\rangle = \sum_{\lambda} \int d\mu(\mathbf{k}) f^{(1)}(\vec{\mathbf{k}}, \lambda) |\vec{\mathbf{k}}, \lambda\rangle$$

$$\langle f^{(1)} | f^{(1)} \rangle = \sum_{\lambda} \int d\mu(\mathbf{k}) |f^{(1)}(\vec{\mathbf{k}}, \lambda)|^2 = 1$$

$$d\mu(\mathbf{k}) |f^{(1)}(\vec{\mathbf{k}}, \lambda)|^2 = \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})} |f^{(1)}(\vec{\mathbf{k}}, \lambda)|^2 =$$

= probability to find photon with momentum $\vec{\mathbf{k}}$ (within $d^3\mathbf{k}$) and polarization λ

$$f^{(1)}(\vec{\mathbf{k}}, \lambda) = \langle \vec{\mathbf{k}}, \lambda | f^{(1)} \rangle = \langle 0 | a(\vec{\mathbf{k}}, \lambda) | f^{(1)} \rangle$$

projector on one-particle subspace

$$P^{(1)} = \sum_{\lambda} \int d\mu(\mathbf{k}) |\vec{\mathbf{k}}, \lambda\rangle \langle \vec{\mathbf{k}}, \lambda|$$

general form of a two-photon state

$$|f^{(2)}\rangle = \sum_{\lambda_1, \lambda_2} \frac{1}{2!} \int d\mu(\vec{k}_1) d\mu(\vec{k}_2) f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2) \underbrace{|\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2\rangle}_{a(\vec{k}_1, \lambda_1)^\dagger a(\vec{k}_2, \lambda_2)^\dagger |0\rangle}$$

$$f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2) = f^{(2)}(\vec{k}_2, \lambda_2; \vec{k}_1, \lambda_1)$$

(note that $\sum_{\lambda_1, \lambda_2} \int d\mu(\vec{k}_1) d\mu(\vec{k}_2)$ as well as

$a(\vec{k}_1, \lambda_1)^\dagger a(\vec{k}_2, \lambda_2)^\dagger$ symmetric under

$$(\vec{k}_1, \lambda_1) \leftrightarrow (\vec{k}_2, \lambda_2)$$

exercise: $\langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | f^{(2)} \rangle = f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)$

$$\langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | f^{(2)} \rangle = \sum_{\lambda'_1, \lambda'_2} \frac{1}{2!} \int d\mu(\vec{p}_1) d\mu(\vec{p}_2) f^{(2)}(\vec{p}_1, \lambda'_1; \vec{p}_2, \lambda'_2)$$

$$\times \langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | \vec{p}_1, \lambda'_1; \vec{p}_2, \lambda'_2 \rangle$$

$$\langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | \vec{p}_1, \lambda_1'; \vec{p}_2, \lambda_2' \rangle =$$

$$= \langle 0 | a(\vec{k}_1, \lambda_1) a(\vec{k}_2, \lambda_2) \underbrace{a(\vec{p}_1, \lambda_1')^\dagger a(\vec{p}_2, \lambda_2')^\dagger}_{a(\vec{p}_1, \lambda_1')^\dagger a(\vec{k}_2, \lambda_2) + \delta(\vec{k}_2, \vec{p}_1) \delta_{\lambda_2, \lambda_1'}} | 0 \rangle$$

$$= \delta(\vec{k}_1, \vec{p}_1) \delta_{\lambda_1, \lambda_1'} \delta(\vec{k}_2, \vec{p}_2) \delta_{\lambda_2, \lambda_2'}$$

$$+ \delta(\vec{k}_1, \vec{p}_2) \delta_{\lambda_1, \lambda_2'} \delta(\vec{k}_2, \vec{p}_1) \delta_{\lambda_2, \lambda_1'}$$

$$\Rightarrow \langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | f^{(2)} \rangle =$$

$$= \frac{1}{2} [f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2) + f^{(2)}(\vec{k}_2, \lambda_2; \vec{k}_1, \lambda_1)]$$

$$\stackrel{\uparrow}{=} f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)$$

$f^{(2)}$ symmetric

two-photon wave function
in momentum space

Bose-symmetry ✓

exercise: normalization

$$1 = \langle f^{(2)} | f^{(2)} \rangle = \frac{1}{4} \sum_{\substack{\lambda_1, \lambda_2 \\ \lambda'_1, \lambda'_2}} \int d\mu(k_1) d\mu(k_2) d\mu(p_1) d\mu(p_2)$$

$$f(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)^* f(\vec{p}_1, \lambda'_1; \vec{p}_2, \lambda'_2)$$

$$\langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | \vec{p}_1, \lambda'_1; \vec{p}_2, \lambda'_2 \rangle$$

$$= \frac{1}{2} \sum_{\lambda_1, \lambda_2} \int d\mu(k_1) d\mu(k_2) |f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)|^2$$

$$|f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)|^2 d\mu(k_1) d\mu(k_2) = \text{probability}$$

to find a photon with polarization λ_1 in

the momentum range $[k_{1x}, k_{1x} + dk_{1x}] \times [k_{1y}, k_{1y} + dk_{1y}] \times$
 $\times [k_{1z}, k_{1z} + dk_{1z}]$ and a second photon

with polarization λ_2 in the momentum range

$$[k_{2x}, k_{2x} + dk_{2x}] \times [k_{2y}, k_{2y} + dk_{2y}] \times [k_{2z}, k_{2z} + dk_{2z}]$$

$$\begin{aligned} & \frac{1}{2} \left[|f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)|^2 d\mu(k_1) d\mu(k_2) \right. \\ & \quad \left. + |f^{(2)}(\vec{k}_2, \lambda_2; \vec{k}_1, \lambda_1)|^2 d\mu(k_2) d\mu(k_1) \right] \\ & = |f^{(2)}(\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2)|^2 d\mu(k_1) d\mu(k_2) \end{aligned}$$

$$\begin{aligned} |f^{(2)}\rangle &= \frac{1}{2!} \sum_{\lambda_1, \lambda_2} \int d\mu(k_1) d\mu(k_2) |\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2\rangle \times \\ & \quad \times \langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2 | f^{(2)} \rangle \end{aligned}$$

$\Rightarrow$ projector on two-particle subspace

$$P^{(2)} = \frac{1}{2!} \sum_{\lambda_1, \lambda_2} \int d\mu(k_1) d\mu(k_2) |\vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2\rangle \langle \vec{k}_1, \lambda_1; \vec{k}_2, \lambda_2|$$

general n-photon state

$$\begin{aligned} |f^{(n)}\rangle &= \sum_{\lambda_1, \dots, \lambda_n} \int \frac{d\mu(k_1) \dots d\mu(k_n)}{n!} f(\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n) \\ & \quad \underbrace{|\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n\rangle}_{a(\vec{k}_1, \lambda_1)^\dagger \dots a(\vec{k}_n, \lambda_n)^\dagger |0\rangle} \end{aligned}$$

n -photon wave function $f^{(n)}(\vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n)$

symmetric under $(\vec{k}_i, \lambda_i) \leftrightarrow (\vec{k}_j, \lambda_j)$

exercises:

$$\langle \vec{k}_1, \lambda_1; \dots, \vec{k}_n, \lambda_n | f^{(n)} \rangle = f^{(n)}(\vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n)$$

$$\begin{aligned} \langle f^{(n)} | f^{(n)} \rangle &= 1 \Leftrightarrow \frac{1}{n!} \sum_{\lambda_1, \dots, \lambda_n} \int d\mu(\vec{k}_1) \dots d\mu(\vec{k}_n) |f^{(n)}(\vec{k}_1, \lambda_1, \dots)|^2 \\ &= 1 \end{aligned}$$

$$\langle \vec{p}_1, \sigma_1; \dots, \vec{p}_n, \sigma_n | \vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n \rangle =$$

$$= \sum_{\pi \in S_n} \prod_{i=1}^n \delta(p_i, k_{\pi(i)}) \delta_{\sigma_i, \lambda_{\pi(i)}}$$

↑
permutation of n elements

$$\Rightarrow P^{(n)} = \frac{1}{n!} \sum_{\lambda_1, \dots, \lambda_n} \int d\mu(\vec{k}_1) \dots d\mu(\vec{k}_n) |\vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n\rangle \langle \vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n|$$

$$\Rightarrow \mathbb{1} = \sum_{n=0}^{\infty} P^{(n)} \quad (P^{(0)} = |0\rangle \langle 0|)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\lambda_1, \dots, \lambda_n} \prod_{i=1}^n d\mu(\vec{k}_i) |\vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n\rangle \langle \vec{k}_1, \lambda_1; \dots \vec{k}_n, \lambda_n|$$

angular momentum

$$J_i = \int d^3x \varepsilon_{ijk} x_j (\vec{E} \times \vec{B})_k$$

$$\begin{aligned} &= \underbrace{\int d^3x \varepsilon_{ijk} x_j E_l \nabla_k A_l}_{L_i} + \underbrace{\int d^3x \varepsilon_{ijk} E_j A_k}_{S_i} \\ &\uparrow \\ &\text{partial integration, } \text{div } \vec{E} = 0 \end{aligned}$$

notation " $\vec{L}$, $\vec{S}$ " should not be taken too seriously:

not gauge-invariant, angular momentum relations

not fulfilled separately

one-photon momentum eigenstate $|\vec{p}, \lambda\rangle = a(\vec{p}, \lambda)^\dagger |0\rangle$

not eigenstate of $\vec{J}$ ($[J_i, P_j] \neq 0$)

but with projection on $\vec{p}/|\vec{p}|$:

$$\vec{L} \cdot \vec{p} |\vec{p}, \alpha\rangle = 0 \Rightarrow \vec{J} \cdot \vec{p} |\vec{p}, \alpha\rangle = \vec{S} \cdot \vec{p} |\vec{p}, \alpha\rangle$$

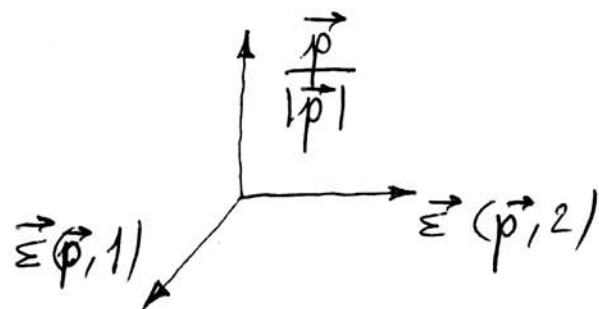
$$\vec{S} = -i \sum_{\lambda, \sigma} \int d\mu(k) \vec{\varepsilon}(\vec{k}, \lambda)^* \times \vec{\varepsilon}(\vec{k}, \sigma) a(\vec{k}, \lambda)^\dagger a(\vec{k}, \sigma)$$

(exercise)

helicity operator $\vec{J} \cdot \vec{P} / |\vec{P}|$

$$\begin{aligned}
 \vec{J} \cdot \vec{p} |\vec{p}, \alpha\rangle &= \\
 &= -i \sum_{\lambda, \sigma} \int d\mu(k) \vec{p} \cdot [\vec{\epsilon}(\vec{k}, \lambda)^* \times \vec{\epsilon}(\vec{k}, \sigma)] \\
 &\quad \times \underbrace{a(\vec{k}, \lambda)^\dagger a(\vec{k}, \sigma)}_{\delta(k, p)} \underbrace{a(\vec{p}, \alpha)^\dagger}_{\delta\sigma\alpha} |0\rangle \\
 &= -i \sum_{\lambda} \vec{p} \cdot [\vec{\epsilon}(\vec{p}, \lambda)^* \times \vec{\epsilon}(\vec{p}, \alpha)] |\vec{p}, \lambda\rangle
 \end{aligned}$$

consider linearly polarized states (real polarization vectors):



$$\vec{J} \cdot \frac{\vec{p}}{|\vec{p}|} |\vec{p}, 1\rangle = i |\vec{p}, 2\rangle$$

$$\vec{J} \cdot \frac{\vec{p}}{|\vec{p}|} |\vec{p}, 2\rangle = -i |\vec{p}, 1\rangle$$

matrix elements of helicity operator
with respect to $|\vec{p}, \alpha\rangle$, $\alpha = 1, 2$:

$$\left(\langle \vec{p}, \alpha | \vec{J} \cdot \frac{\vec{p}}{|\vec{p}|} | \vec{p}, \beta \rangle \right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

eigenvalues ± 1

diagonalization $\rightarrow$ circularly polarized
one-photon states (helicity eigenstates)

$$|\vec{p}, +\rangle = -\frac{1}{\sqrt{2}} (|\vec{p}, 1\rangle + i |\vec{p}, 2\rangle)$$

$$|\vec{p}, -\rangle = \frac{1}{\sqrt{2}} (|\vec{p}, 1\rangle - i |\vec{p}, 2\rangle)$$

$$\Rightarrow a(\vec{p}, \lambda)^\dagger = -\frac{\lambda}{\sqrt{2}} [a(\vec{p}, 1)^\dagger + i\lambda a(\vec{p}, 2)^\dagger]$$

$$\lambda = \pm$$

$$\Rightarrow a(\vec{p}, \lambda) = -\frac{\lambda}{\sqrt{2}} [a(\vec{p}, 1) - i\lambda a(\vec{p}, 2)]$$

remark: consistent with our convention

$$\vec{\epsilon}(\vec{p}, \lambda) = -\lambda \frac{\vec{\epsilon}(\vec{p}, 1) + i\lambda \vec{\epsilon}(\vec{p}, 2)}{\sqrt{2}}$$